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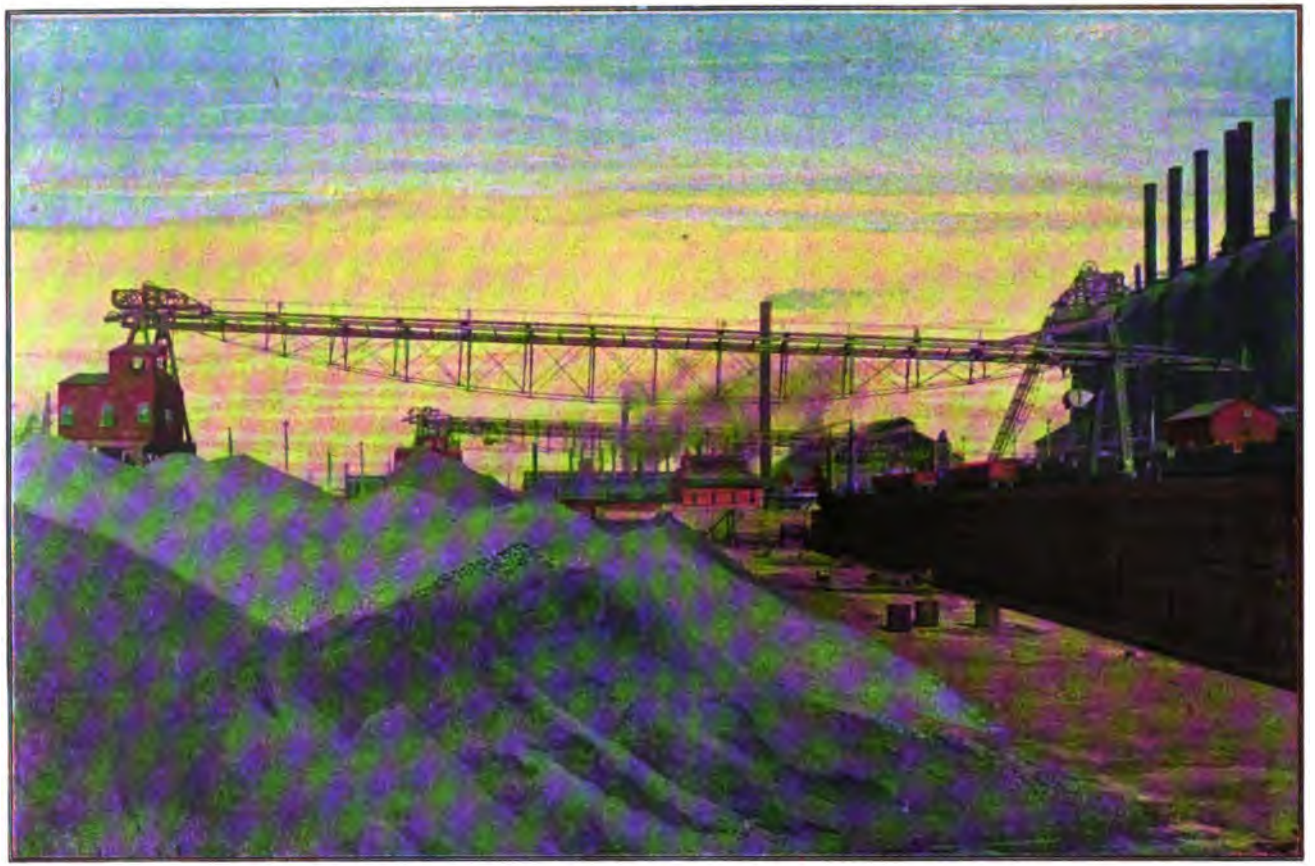

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VOLUME II.

MATHEMATICS-MECHANICS-HEAT

AMERICAN SCHOOL of CORRESPONDENCE at ARMOUR INSTITUTE of TECHNOLOGY

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Volume II.

MATHEMATICS MECHANICS HEAT

SECTION I.

PLANE GEOMETRY.

INTRODUCTORY DEFINITIONS.

1. **Mathematics** is the science of quantity.

2. **Quantity** is that which can be measured; as distance, time, weight.

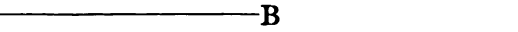
3. **Geometry** is that branch of mathematics which treats of the properties of extension.

4. **Extension** has one or more of the three dimensions, length, breadth, or thickness.

5. **A Point** has position, but not magnitude.

6. **A Line** has length, without breadth or thickness.

7. **A Straight Line** is one whose direction is the same throughout, as A B:



A straight line has two directions exactly opposite, of which either may be assumed as its direction.

(NOTE. The word line used alone in this book, means a straight line.)

8. **Corollary.** Two points of a line determine its position.

9. **A Curved Line** is one whose direction is constantly changing, as C D:

10. **A Surface** has length and breadth, but no thickness.

11. **A Plane** is such a surface that a straight line joining any two of its points is wholly in the surface.

12. **A Solid** has length, breadth, and thickness.

13. The boundaries of solids are surfaces; of surfaces, lines; the ends of lines are points.

14. **A Theorem** is a truth which requires to be proved.

15. **A Problem** is a question proposed for solution.

16. **A Proposition** is a general name for either a theorem or a problem.

17. A **Corollary** is an inference from a proposition or statement.

18. A **Scholium** is a remark appended to a proposition.

19. A **Hypothesis** is a supposition in the statement of a proposition, or in the course of a demonstration.

20. An **Axiom** is a self-evident truth.

AXIOMS.

1. If equals be added to equals, the sums are equal.

2. If equals are subtracted from equals, the remainders are equal.

3. If equals are multiplied by equals, the products are equal.

4. If equals are divided by equals, the quotients are equal.

5. Like powers and like roots of equals are equal.

6. The whole of a magnitude is greater than any of its parts.

7. The whole of a magnitude is equal to the sum of all its parts.

8. Magnitudes respectively equal to the same magnitude are equal to each other.

9. A straight line is the shortest distance between two points.

ABBREVIATIONS.

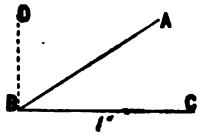
The following list of abbreviations will be found useful:

+, plus.	>, is greater than.
—, minus.	<, is less than.
×, multiplied by.	∴, therefore.
=, equals,	∠, angle.
∟, right angle.	∠s, angles.
⊥, perpendicular.	△, triangle.
⊥s, perpendiculars.	△s, triangles.
∥, parallel.	▱, parallelogram.
∥s, parallels.	▱s, parallelograms.
○, circle.	⌒, arc.
○s, circles.	≍, equivalent to.

ANGLES, LINES, POLYGONS.

GENERAL PRINCIPLES AND DEFINITIONS.

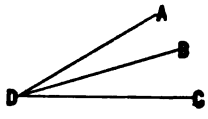
21. When two straight lines meet at a point, the difference in their direction is an *angle*. The point of meeting is called the *vertex*, and the lines the *sides*, of the angle. Thus in the angle formed by the lines A B and B C, B is the vertex, and A B and B C are the sides.



22. In speaking of an angle at a given vertex, it can be designated by the letter at that vertex, if, as here, there is but one angle at that vertex. But when two or more angles have a common vertex, then it is necessary, to avoid ambiguity, to name also the letters at the extremities of the sides, placing the letter at the vertex in the middle. Thus we say here "angle B"; but if D B be drawn, we should have to say "angle A B C," or "angle C B A" to name the same angle before designated.

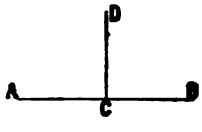
23. The size of an angle depends solely upon the amount of divergence of its sides, not at all upon their length.

24. When two angles have the same vertex and a side common to both, they are called *adjacent* angles; as angle A D B, and angle B D C.

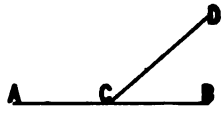


The side D B is common to both.

25. If a straight line meets another so as to make the adjacent angles equal, each of these angles will be a right angle; and the two lines will be perpendicular to each other. Thus angle A C D and angle D C B, being equal, are right angles, and A B and D C are perpendicular to each other.



26. **An Acute Angle** is less than a right angle; as angle B C D.



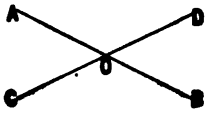
27. An obtuse angle is greater than a right angle; as angle A C D.

Acute and obtuse angles are called *oblique* angles.

28. If the sum of two angles is equal to a right angle, or 90°, each is called the *complement* of the other; if their sum is equal to two right angles, 180°, they are called the *supplements* of each other.

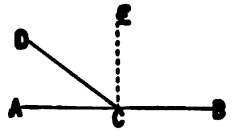
It is evident that complements of equal angles are equal to each other; and the same is true of their supplements.

29. Two angles are called *vertical* or *opposite*, when the sides of one are the prolongations of the sides of the other, as angle A O C and angle B O D.



THEOREM I.

30. *If one straight line meets another straight line, the adjacent angles thus formed, are equal to two right angles.*



Let D C meet the straight line A B at the point C.

To prove that A C D + D C B = two right angles.

At C, erect the perpendicular C E; then it is evident that

A C D + D C E = A C E = one right angle.

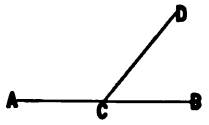
And B C D − D C E = B C E = one right angle

Adding we have

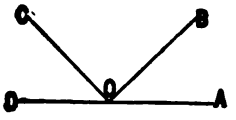
A C D + B C D = A C E + B C E = two right angles.

31. *Corollary* 1. If only two angles are formed, each is the supplement of the other. For by the Theorem,

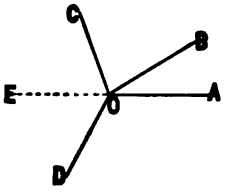
A C D + B C D = 2 right angles.
∴ A C D = 2 right angles — D C B,
or D C B = 2 right angles — A C D.



32. *Corollary* 2. The sum of all the angles A O B, B O C, C O D, formed on the same side of a straight line at a given point = two right angles. For, by the Theorem, A O B + B O D = two right angles, and this sum is equal to the sum of the angles A O B, B O C and C O D.



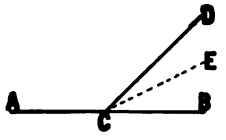
33. *Corollary* 3. The sum of all the angles A O B, B O C, C O D and D O A formed about a point is equal to four right angles. For if the line A O be produced to E, by the preceding corollary, the sum of the angles A O B, B O C and C O E = two right angles, and the same is true of the sum of the angles A O D and D O E.



Hence, the sum of the angles A O B, B O C, C O D, and D O A = four right angles.

THEOREM II.

34. ***If at a point in a straight line two other straight lines upon opposite sides of it make the sum of the adjacent angles equal to two right angles, these two lines form a straight line.***



Let the sum of the adjacent angles A C D and B C D be equal to two right angles.

To prove that A C B is a straight line.

If A C B is not a straight line, let C E be in the same straight line with A C.

By Corollary 1, angle E C D is the supplement of angle A C D.

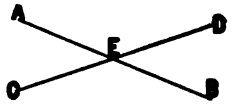
But by hypothesis angle B C D is the supplement of A C D.

Therefore angle B C D = angle E C D, which is impossible unless C E coincides with C B.

Hence C B is in the same straight line with A C.

THEOREM III.

35. ***If two straight lines cut each other, the vertical angles are equal.***



Let the straight lines A B and C D intersect at E.

To prove that angle A E C = angle B E D.

By § 28, angle A E C is the supplement of angle A E D.

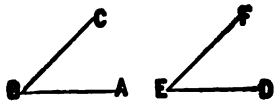
Angle B E D is also the supplement of angle A E D; and

∴ angle A E C = angle B E D. (Also by § 28.)

In like manner it may be proved that angle A E D = angle B E C.

THEOREM IV.

36. ***Two angles whose sides have the same or opposite directions are equal.***

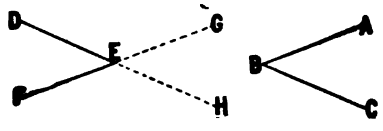


1st, Let B A and B C, including angle B, have respectively the same directions as E D and E F, including the angle E;

To prove angle B = angle E.

Since B A has same direction as E D, and B C has same direction as E F, the difference of direction of B A and B C must be the same as the difference of direction of E D and E F.

∴ angle B = angle E.

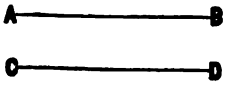


2nd, Let B A and B C, including angle B, have respectively opposite directions to E F and E D, including angle E;

To prove angle B = angle E.
Produce D E and F E so as to form angle G E H;
Then, angle G E H = angle D E F (Theorem III),
And, angle G E H = angle A B C (By hypothesis).
∴ angle B = angle E.

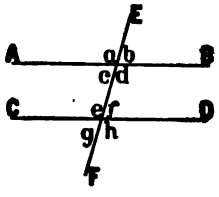
PARALLEL LINES.

37. *Definition.* Two straight lines are called parallel when they lie in the same plane and cannot meet however far they may be produced, as A B and C D.



38. *Axiom.* Two lines parallel to a third are parallel to each other.

39. *Definitions.* If two straight lines A B and C D are cut by a third line E F, the eight angles thus formed are named as follows:



The angles *c*, *d*, *e*, *f* are called *interior* angles.

The angles *a*, *b*, *g*, *h* are called *exterior* angles.

The angles *c* and *f* or *d* and *e* are called *alternate-interior* angles.

The angles *a* and *h* or *b* and *g* are called *alternate-exterior* angles.

The angles *a* and *e*, *b* and *f*, *c* and *g*, or *d* and *h* are called *corresponding* angles.

THEOREM V.

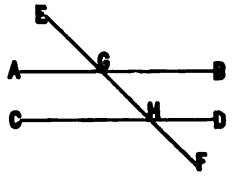
40. *If two parallels are cut by a third straight line:*

1st. All the corresponding angles are equal.

2nd. The alternate-interior angles are equal.

3rd. The interior (as also the exterior) angles on the same side (of the intersecting line E F*), are supplements of each other.*

Let E F cut the parallels A B and C D. To prove:



1st. The corresponding angles E G A and G H C, or E G B and G H D, are equal, since their sides have the same direction. (Theorem IV).

2nd. The alternate interior angles A G H and G H D, or B G H and G H C, are equal, because their sides have opposite directions. (Theorem IV).

3rd. The internal angles on the same side, A G H and G H C, or B G H and G H D, are supplements of each other; for angle A G H is the supplement of angle A G E (corollary of Theorem I), which has been proved equal to angle G H C. In the same way it may be proved that angles B G H and G H D are supplements of each other.

THEOREM VI.

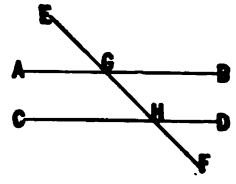
41. *Conversely if a straight line cut two other straight lines in the same plane, these two lines are parallel.*

I. If the corresponding angles are equal.

II. If the alternate interior angles are equal.

III. If the interior, as also the exterior, angles on the same side (of the intersecting line) are supplements of each other.

Let E F cut the two lines A B and C D so as to make angle E G B = angle G H D, or angle A G H = angle G H D, or angles B G H and G H D supplements of each other.



To prove that A B is parallel to C D:

If through the point G a line is drawn parallel to C D, it will make the correspond-

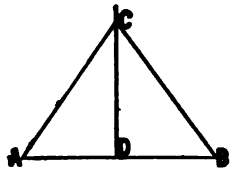
ing angles equal, and the alternate interior angles equal, and the interior angles, as also the exterior, on the same side (of the intersecting line) supplements of each other, by (Theorem V); therefore it must coincide with A B; that is, A B is parallel to C D.

THEOREM VII.

42. *If a perpendicular is erected at the middle point of a straight line.*

I. Any point in the perpendicular is equally distant from the extremities of the line.

II. Any point without the perpendicular is unequally distant from the extremities of the line.



I. Let C D be perpendicular to A B at its middle point D, and let C be any point in C D.

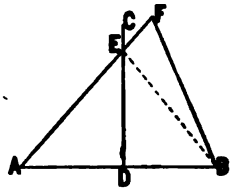
To prove that C is equally distant from A and B:

Join A C and B C.

Let the figure B D C be superposed upon A D C by folding it over about D C as an axis.

Since angle B D C = angle A D C (adjacent angles, formed by a perpendicular meeting another line), and B D = A D (by hypothesis), the point B will fall at A and B C will coincide with A C.

Therefore B C = A C, and C is equally distant from A and B.



II. Let C D be perpendicular to A B at its middle point D, and let F be any point without C D.

To prove that F is unequally distant from A and B.

Join A F and B F; let A F intersect C D at E, and draw B E.

Since the straight line B F is shorter than the broken line B E F (a straight line being the shortest distance between two points),

B F is less than B E + E F.

Now we have already proved that B E = A E, (the first part of this theorem.)

Hence B F is less than A E + E F, or B F is less than A F.

Therefore F is unequally distant from A and B.

43. *Corollary* 1. When the figure B D E is superposed upon A D E, the angles E B D and B E D coincide with the angles E A D and A E D, respectively: that is,

angle E A D = angle E B D, and angle A E D = angle B E D.

Hence, *if straight lines are drawn to the extremities of a straight line from any point in the perpendicular erected at its middle point, they make equal angles with the line and with the perpendicular.*

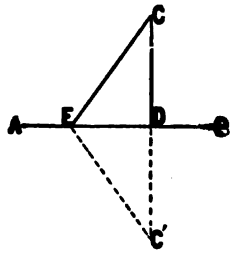
44. *Corollary* II. If a series of points, all of which satisfy a given condition, lie in a certain line, that line is called the *locus* of the points. For example, all the points which satisfy the condition of being equally distant from the extremities of a straight line lie in the perpendicular erected at its middle point.

In other words, *the perpendicular at the middle point of a straight line is the locus of points which are equally distant from the extremities of the line.*

45. *Scholium.* We know by *axiom* that a straight line is determined by any two of its points. It therefore follows that *two points equally distant from the extremities of a line determine its perpendicular at its middle point.*

THEOREM VIII.

46. *The perpendicular is the shortest line which can be drawn from a point to a straight line.*



Let C D be the perpendicular from C to A B, and C E any oblique line.

To prove that C D is less than C E.

Produce C D to C′, making C′ D = C D, and join E C′.

Then since E D is perpendicular to C C′ at its middle point, C E = C′ E, (Theorem VII.)

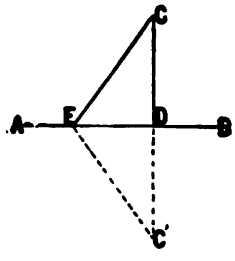
But the straight line C C′ being shorter than the broken line C E C′, we have

C D + C′ D is less than C E + C′ E

Therefore, 2 C D is less than 2 C E, or C D is less than C E.

THEOREM IX.

47. *From a given point outside a straight line, but one perpendicular can be drawn to that line.*



Let C be the given point outside the line A B, and draw C D perpendicular to A B.

To prove that C D is the only perpendicular which can be drawn.

If possible, let C E be another perpendicular from C to A B.

Produce C D to C′, making C′ D = C D, and then join C′ E.

Then since E D is perpendicular to C C′ at its middle point, angle C′ E D = angle C E D (§ 43.)

But C E D is a right angle (by hypothesis.)

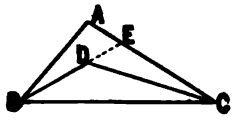
Hence C′ E D is also a right angle, and the sum of the adjacent angles C E D and C′ E D is equal to two right angles, and therefore C E C′ is a straight line (Theorem II.)

Now the latter is impossible, since but one straight line can be drawn between two given points (*Axiom.*)

Hence C E cannot be perpendicular to A B, and C D is the only perpendicular which can be drawn.

THEOREM X.

48. *If two lines are drawn from a point to the extremities of a straight line, their sum is greater than the sum of two other lines similarly drawn, but enveloped by them.*



Let A B and A C be drawn from the point A to the extremities of the line B C, and let D B and D C be two lines similarly drawn but enveloped by A B and A C.

To prove that A B + A C is greater than D B + D C:

Produce B D to meet A C at E.

Then since the broken line B A E is greater than the straight line B E, the line B A C is greater than B E C.

And since the broken line D E C is greater than the straight line D C, the line B E C is greater than B D C.

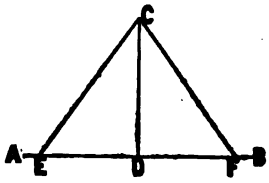
Therefore, A B + A C is greater than D B + D C.

THEOREM XI.

49. *If oblique lines are drawn from a point to a straight line.*

I. Two oblique lines cutting off equal distances from the foot of the perpendicular are equal.

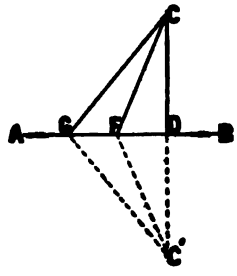
II. Of two oblique lines cutting off unequal distances from the foot of the perpendicular, the more remote is the greater.



I. Let the oblique lines C E and C F meet the line A B at equal distances from the foot of the perpendicular C D.

To prove that C E = C F.

Since C D is perpendicular to E F at its middle point D, C E = C F (Theorem VII.)



II. Let the oblique lines C F and C G meet the line A B at unequal distances from the foot of the perpendicular C D, and let C G be the more remote.

To prove that C G is greater than C F:

Produce C D to C′, making C′ D = C D, and then join C′ F and C′ G.

Now C F = C′ F, and C G = C′ G. (Theorem VII).

But the broken line C G C′ is greater than the broken line C F C′, which is enveloped by it. (Theorem X).

Hence C G + C′ G is greater than C F + C′ F.

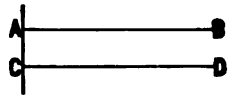
Therefore 2 C G is greater than 2 C F, or C G is greater than C F.

50. *Corollary I.* Conversely two equal oblique lines from a point to a straight line, cut off equal distances from the foot of the perpendicular.

51. *Corollary II.* Only two equal straight lines can be drawn from the same point to the same straight line.

THEOREM XII.

52. *Two straight lines perpendicular to the same straight line are parallel.*



Let the lines A B and C D be perpendicular to A C.

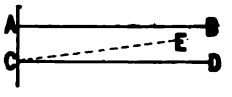
To prove that they are parallel:

If they are not parallel, they will meet if sufficiently produced (*definition*), and we should then have two perpendiculars from the same point to the same straight line, which is impossible. (Theorem IX).

Therefore, they cannot meet and are parallel.

THEOREM XIII.

53. *A straight line perpendicular to one of two parallels is perpendicular to the other.*



Let A B and C D be parallel lines, and let A C be perpendicular to A B.

To prove that A C is also perpendicular to C D:

Let C E be perpendicular to A C.

Then C E is parallel to A B. (Theorem XII).

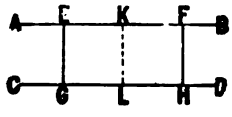
But C D is parallel to A B (*by hypothesis*).

Hence C E must coincide with C D, since but one straight line can be drawn through the point C parallel to A B (*by axiom*).

Therefore A C is perpendicular to C D.

THEOREM XIV.

54. *Two parallel lines are everywhere equally distant.*



Let A B and C D be two parallel lines, and let E and F be any two points on A B.

To prove that E and F are equally distant from C D:

Let K be the middle point of E F, and draw E G, K L, and F H perpendicular to A B and C D. (Theorem XIII).

Let the portion of the figure to the right of K L be folded over about K L as an axis until it comes into the plane of the figure on the left.

Since angle F K L = angle E K L (right angles by construction), and F K = E K (by construction), the line F K will fall upon E K and the point F will fall upon E. And since, in the same way, angle K L D = angle K L C, the line D L will fall upon C L.

Therefore the line F H, which is perpendicular to D L will coincide with E G, which is perpendicular to C L (Theorem IX), and the point H will fall at G.

Hence E G = F H and the points E and F are equally distant from C D.

PLANE FIGURES.

DEFINITIONS.

55. **A Plane Figure** is a portion of a plane bounded by lines, either curved or straight.

If the bounding lines are straight the figure is a **polygon**, and the sum of the bounding lines is the **perimeter**.

56. An **Equilateral Polygon** is one whose sides are equal each to each.

57. An **Equiangular Polygon** is one whose angles are equal each to each.

58. Polygons whose sides or angles are respectively equal are called *mutually* equilateral or mutually equiangular.

Two equal sides, or two equal angles, one in each polygon, similarly situated, are called *homologous* sides or angles.

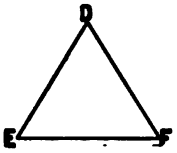
59. Polygons are spoken of as **equal** only when, being applied to each other, they exactly coincide.

60. Of Polygons, the simplest has three sides, and is called a *triangle;* one of four sides is a *quadrilateral;* one of five, a *pentagon:* one of six, a *hexagon*; one of eight, an *octagon;* one of ten a *decagon.*

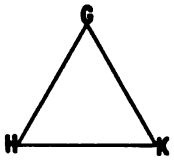
TRIANGLES.

61. When a triangle has no two of its sides equal, it is termed *scalene;* as A B C.

62. When a triangle has two of its sides equal, it is termed *isosceles*; as D E F.

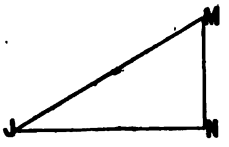


63. When a triangle has all its sides equal, it is termed *equilateral;* as G H K.

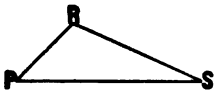


64. When one of the angles of a triangle is a right angle, the triangle is termed a *right triangle*, as J M N.

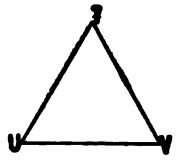
The side J M opposite the right angle is termed the *hypothenuse.*



65. When one of the angles of a triangle is an obtuse angle, the triangle is termed an *obtuse-angled triangle:* as P R S.



66. When all the angles of a triangle are acute angles, the triangle is termed an *acute-angled triangle;* as T U V.



67. The side upon which any polygon is supposed to be resting is usually called its *base;* but is an isosceles triangle, as D E F above, in which DE = D F, the third side E F is always considered the base.

When any side has been taken as *base*, the opposite angle is called the *vertical angle*, and its vertex is called the vertex of the triangle.

The *altitude* of a triangle is the perpendicular drawn from the vertex to the base.

68. Since a straight line is the shortest distance between two points, it follows that *either side of a triangle is less than the sum of the other two.*

THEOREM XV.

69. *Either side of a triangle is greater than the difference of the other two.*

Let A B C be any triangle.

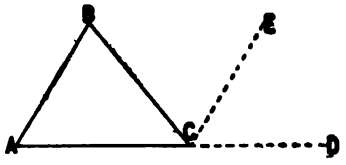
To prove that any side as A B, is greater than the difference of the other two:

Since a straight line is the shortest distance between two points, A B + A C is greater than B C.

Subtracting A C from both members of the inequality, A B is greater than B C — A C.

THEOREM XVI.

70. *The sum of the angles of any triangle is equal to two right angles.*



Let A B C be any triangle.

To prove that the sum of angles A, B, and C is equal to two right angles:

Produce A C to D and draw C E parallel to A B; then angle D C E = angle A (corresponding angles); angle B C E = angle B (alternate angles). Hence angles D C E + B C E + B C A = A + B + B C A, but angles D C E + B C E + B C A = two right angles. (Theorem I).

Therefore angles A + B + B C A = two right angles.

71. *Corollary I.* If any side of a triangle is produced, the exterior angles thus formed is equal to the sum of the two opposite interior angles.

73. *Corollary II.* If two triangles have two angles of the one respectively equal to two angles of the other, the remaining angles are equal.

74. *Corollary III.* A triangle cannot have two right or two obtuse angles; for in either case the sum of the three angles in the triangle would exceed two right angles.

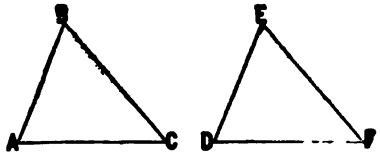
75. *Corollary IV.* In a right triangle the sum of the two acute angles is equal to a right angle.

76. *Corollary V.* If two angles of a triangle are known, the third can be formed by subtracting their sum from two right angles.

Corollary VI. An exterior angle of a triangle is equal to the sum of the two opposite interior angles, as angle B C D = angle A + angle B.

THEOREM XVII.

77. *If two triangles have two sides and the included angle of the one equal respectively to two sides and the included angle of the other, the two triangles are equal in all respects.*



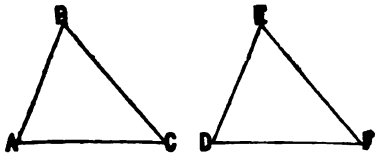
In the triangles A B C and D E F, let the side A B = D E, A C = D F and the angle A = angle D.

To prove triangle A B C = triangle D E F:

Superpose A B C upon D E F, so that A B coincides with its equal D E; then as angle A = angle D, and as A C = D F, A C will take the direction of D F and will coincide with it. Therefore the extremities of the third lines B C and E F respectively coinciding, the lines themselves must do so, and the two triangles are proved to be equal in all respects.

THEOREM XVIII.

78. *If two triangles have a side and the two adjacent angles of the one equal respectively to a side and the two adjacent angles of the other, the two triangles are equal.*



In the triangles A B C and D E F, let angle A = angle D, angle C = angle F, and the side A C = D F.

To prove that triangle A B C is equal in all respects to triangle D E F:

Place the side A C on its equal D F, with the point A on the point D, and the point C on the point F, as A C = D F (by hyp.); then since angle A = angle D, A B will take the direction of

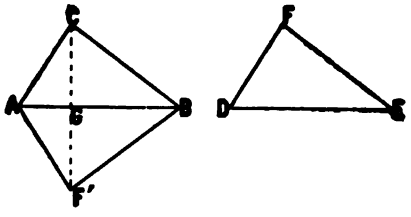
D E, and since angle C = angle F, C B will take the direction of F E; and the point B falling at once in each of the lines D E and F E must be at their point of intersection E; therefore the two triangles coincide and are equal.

79. *Corollary I.* Two right triangles are equal when the hypothenuse and an acute angle of one are respectively equal to the hypothenuse and an acute angle of the other; for the other acute angles are equal (Theorem XVI, Corollary II, and the triangles are then equal by Theorem XVIII).

Corollary II. Two right triangles are equal when a side and an acute angle of one are respectively equal to a side and an acute angle of the other.

THEOREM XIX.

80. ***Two triangles are equal when the three sides of the one are equal respectively to the three sides of the other.***



In the triangles A B C and D E F, let the side A B = D E, B C = E F, and C A = F D.

To prove that the triangles are equal in all respects:

Place the triangle D E F in the position A B F', so that D E shall coincide with its equal A B, the vertex F falling at F' on the opposite side of A B from C.

Draw C F' intersecting A B at G.

Since by hypothesis A C = A F′ and B C = B F′, the the points A and B are equally distant from C and F′.

Hence the line A B is perpendicular to C F′ at its middle point G (Theorem VII).

Let now the triangle A B C be folded over upon A B as axis, until it falls into the plane of A B F′.

Since angle A G C = angle A G F′, and C G = F′ G, the point C will fall at F′.

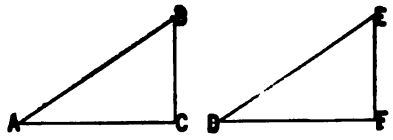
Therefore the triangles coincide throughout and are equal.

Corollary. Since the triangles coincide throughout,

angle A = angle D,
angle B = angle E,
angle C = angle F.

THEOREM XX.

81. ***Two right triangles having the hypothenuse and a side of one respectively equal to the hypothenuse and a side of the other, are equal in all respects.***



In the right triangles A B C and D E F, let the hypothenuse A B be equal to D E, and the side B C equal to E F.

To prove that the triangles are equal:

Superpose the triangle A B C upon D E F so that the side B C shall coincide with its equal E F, the point B falling at E and C at F.

Then since angle C = angle F, the side A C will fall upon D F.

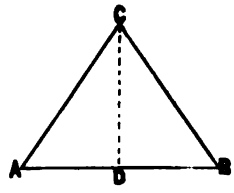
But the equal oblique lines A B and D E cut off upon D F equal distances from the foot of the perpendicular E F (Theorem VII).

This is the point A will fall at D.

Therefore the triangles coincide throughout and are equal.

THEOREM XXI.

82. *In an isosceles triangle the angles opposite the equal sides are equal.*



Let A B C be an isosceles triangle in which A C and B C are the equal sides.

To prove that angle A = angle B:

Draw C D perpendicular to A B.

There in the right triangles A D C and B D C, the side C D is common and A C = B C by hypothesis.

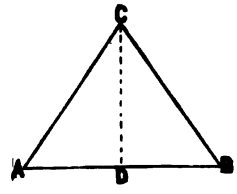
Therefore by Theorem XX, the triangles are equal in all respects, and angle A = angle B.

83. *Corollary I.* From the equality of the triangles A C D and B C D, A D = D B, and the angle A C D = angle B C D; that is, the line bisecting the angle opposite the base of an isosceles triangle bisects the base at right angles, and also bisects the triangle. Also the line drawn from the vertex perpendicular to the base of an isosceles triangle, bisects the base, the vertical angle, and the triangle.

84. *Corollary II.* An equilateral triangle is equiangular.

THEOREM XXII.

85. Conversely, *if two angles of a triangle are equal, the sides opposite them are equal, and the triangle is isosceles.*



In the triangle A B C let the angles A and B be equal.

To prove that A C = B C:

Draw C D perpendicular to A B.

Then in the right triangles A C D and B C D the side C D is common, and by hypothesis angle A = angle B.

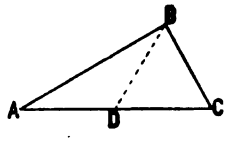
Therefore the triangles are equal in all respects (Theorem XVIII, Corollary II), and A C = B C.

86. *Corollary.* An equiangular triangle is also equilateral.

THEOREM XXIII.

87. *The greater side of a triangle is opposite the greater angle; and conversely, the greater angle is opposite the greater side.*

In the triangle A B C let angle B be greater than C;



To prove that A C is greater than A B:

At the point B make the angle C B D equal to the angle C;

Then D B = D C (Theorem XXII), and A C = A D + D C = A D + D B.

But (by axiom 9) A D + D B is greater than A B.

Therefore A C is greater than A B.

Conversely. Let A C be greater than A B.

NOTE.—In a figure for this proof D would fall nearer C.

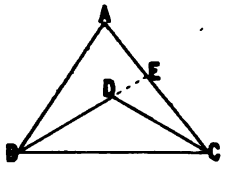
To prove that angle A B C is greater than angle C:

Cut off A D = A B and join B D; then as A D = A B, the angle A B D = angle A D B (Theorem XXI), and angle A D B is greater than angle C (Theorem XVI, Corollary I.)

Therefore angle A B D is greater than angle C; but angle A B C is greater than angle A B D; therefore angle A B C is greater than angle C.

THEOREM XXIV.

88. *If two straight lines are drawn from a point within a triangle to the extremities of any side, the angle included by them is greater than the angle included by the other two sides.*



Let D B and D C be drawn from any point D within the triangle A B C to the extremities of any side, as B C.

To prove that angle B D C is greater than angle A:

Produce B D to meet A C at E.

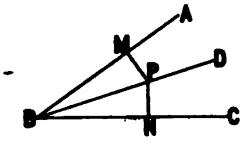
Then the angle B D C, being equal to the sum of the opposite interior angles D E C and D C E (Theorem XVI, Corollary VI), is greater than angle D E C.

And the angle D E C, being equal to the sum of the opposite interior angles A and A B E, is greater than angle A.

Therefore angle B D C is greater than angle A.

THEOREM XXV.

89. *Any point in the bisector of an angle is equally distant from the sides of the angle.*



Let B D be the bisector of an angle, A B C, and let P be any point in B D.

To prove that P is equally distant from A B and B C:

Draw P M and P N perpendicular to A B and B C respectively.

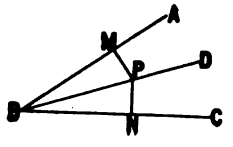
Then in the right triangles P B M and P B N, the side P B is common, and by hypothesis angle P B M = angle P B N.

Therefore the triangles are equal (Theorem XVIII, Corollary I), and P M = P N.

Hence P is equally distant from A B and B C.

THEOREM XXVI.

90. Conversely, *every point which is within an angle, and equally distant from its sides, lies in the bisector of the angle.*



Let P be equally distant from the sides of the angle A B C; join P and B by the line P B.

To prove that P B is the bisector of the angle A B C:

Draw P M and P N, perpendicular respectively to A B, and B C.

Then in the right triangles P B M and P B N, the side P B is common, and by hypothesis P M = P N.

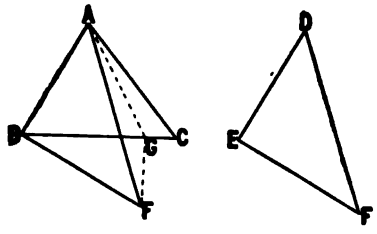
Therefore the triangles are equal, and angle P B M = angle P B N. (Theorem XX).

Hence P B bisects angle A B C.

91. *Scholium.* The bisector of the angle is the *locus* of points which are within the angle and equally distant from its sides.

THEOREM XXVII.

92. ***If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, the third side of the first is greater than the third side of the second.***



In the triangles A B C and D E F, let the side A B be equal to D E, and A C to D F, and let the included angle B A C be greater than angle E D F.

To prove that B C is greater than E F:

Place the triangle D E F in the position A B F′, the side D E coinciding with its equal A B, and the vertex F falling at F′.

Draw A G bisecting the angle C A F′, and join F′ G.

Then in the triangles A C G and A F′ G, the side A G is common, A C = A F′ by hypothesis, and angle C A G = angle F′ A G by construction.

Therefore the triangles are equal in all respects (Theorem XVII), and F′ G = C G.

But by axiom (a straight line being the shortest distance between two points), B G + F′ G is greater than B F′.

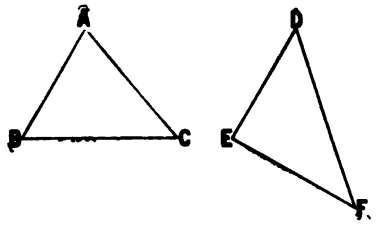
Substituting for F′ G its equal C G,

B G + C G is greater than B F′

That is, B C is greater than E F.

THEOREM XXVIII.

93. Conversely, *if two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, the included angle of the first is greater than the included angle of the second.*



In the triangles A B C and D E F, let the side A B be equal to D E, and A C to D F, and let B C be greater than E F.

To prove that angle A is greater than angle D:

If angle A were equal to angle D, the triangles would be equal (Theorem XVII), and B C would be equal to E F.

And if angle A were less than angle D, B C would be less than E F (Theorem XXVII).

Both of these conclusions are contrary to the hypothesis;

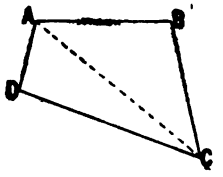
Hence angle A is greater than angle D.

94. *Scholium.* The method of proof employed in Theorem XXVIII, is known as the "Reductis ad absurdum," or "indirect" method. The truth of the theorem is demonstrated by supposing it to be false, and showing that the hypothesis leads to a false conclusion.

QUADRILATERALS.

DEFINITIONS.

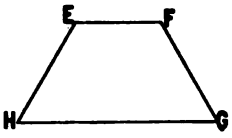
95. A **Quadrilateral** is a plane figure bounded by four straight lines, as A B C D.



96. A **diagonal** of a quadrilateral is a straight line joining two opposite vertices, as A C.

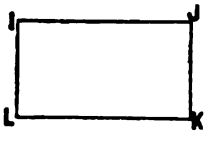
97. A **Trapezium** is a quadrilateral which has no two of its sides parallel; as A B C D.

98. A **Trapezoid** is a quadrilateral which has only two of its sides parallel; as E F G H.

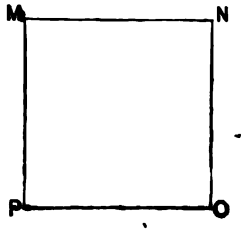


99. A **Parallelogram** is a quadrilateral whose opposite sides are parallel, as any rectangle or rhombus.

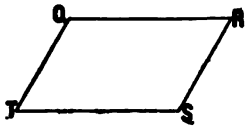
100. A **Rectangle** is a right-angled parallelogram, as I J K L.



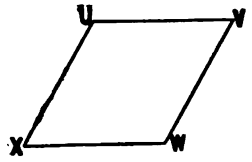
101. A **Square** is an equilateral rectangle, as M N O P.



102. A **Rhomboid** is an oblique-angled parallelogram, as Q R S T.



103. A **Rhombus** is an equilateral rhomboid, as U V W X.



104. The **bases** of a trapezoid are its parallel sides; the **altitude** is the perpendicular distance between them.

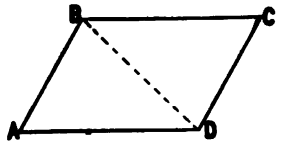
105. The bases of a parallelogram are the side upon which it is supposed to stand and its parallel side; the altitude is the perpendicular distance between them.

THEOREM XXIX.

106. ***The opposite sides and angles of a parallelogram are equal to each other.***

Let A B C D be a parallelogram;

To prove that A B = D C, B C = A D, and the angles A and B respectively equal C and D:



Draw the diagonal B D. In the triangles A B D and B C D, the side B D is common. As B C and A D are parallel, the alternate interior angles C B D, and B D A are equal. (Theorem V.) And in the same way angle A B D = angle B D C.

Therefore the two triangles A B D and B D C are equal (Theorem XVIII;) and the sides opposite the equal angles are equal, i.e. A B = D C, and B C = A D; also the angle A = angle C and the angle

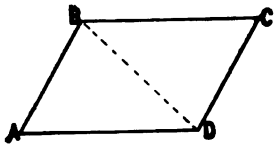
A B C = angle A B D + D B C = B D C + B D A = A D C.

107. *Corollary I.* The diagonal divides a parallelogram into two equal triangles.

108. *Corollary II.* Parallels included between parallels are equal.

THEOREM XXX.

109. *If two sides of a quadrilateral are equal and parallel the figure is a parallelogram.*



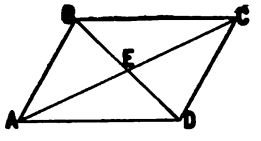
Let A B C D be a quadrilateral having B C equal and parallel to A D;

To prove that A B C D is a parallelogram:

Draw the diagonal B D. As B C is parallel to A D, the alternate interior angles C B D and B D A are equal (Theorem V), therefore the two triangles C B D and B D A are equal (Theorem XVII); and the alternate interior angles A B D and B D C are equal. Therefore A B is parallel to D C (Theorem VI), and A B C D is a parallelogram.

THEOREM XXXI.

110. *The diagonals of a parallelogram bisect each other.*



Let the diagonals of the parallelogram A B C D intersect at E.

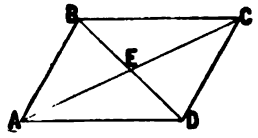
To prove that A E = E C and B E = E D:

In the triangles A E D and B E C, we have A D = B C, (Theorem XXIX), also since the parallels A D and B C are cut by A C, angle E A D = angle E C B. And since the parallels A D and B C are cut by B D, angle E D A = angle E B C.

Therefore the triangles are equal (Theorem XVIII), and A E = E C, and B E = E D (homologous sides of equal triangles).

THEOREM XXXII.

111. Conversely, *if the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*



Let the diagonals of the quadrilateral A B C D bisect each other at E.

To prove that the figure is a parallelogram.

In the triangles A E D and B E C, A E = E C, and D E = E B by hypothesis.

Also angle A E D = angle B E C (vertical angles).

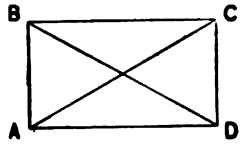
Therefore the triangles are equal (Theorem XVII) and A D = B C, (homologous.)

Similarly the triangles A E B and C E D are equal, and A B = C D, (homologous.)

Hence the opposite sides of the quadrilateral are equal, and the figure is a parallelogram, (Converse of Theorem XXIX, easily proved, by finding that the two triangles formed by the diagonal of such a figure are equal by Theorem XIX; three sides of one equalling three sides of the other.)

THEOREM XXXIII.

112. *The diagonals of a rectangle are equal.*



Let A B C D be a rectangle.

To prove that A C = B D.

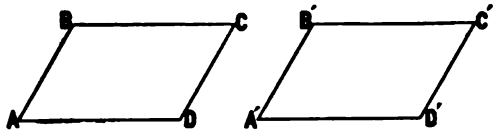
In the right triangles A B C and B C D, the side B C is common, and A B = C D (Theorem XXIX.)

Therefore the triangles are equal (Theorem XVII), and A C = B D (homologous).

113. *Corollary.* The diagonals of a square are equal, and bisect each other at right angles; for the square is at once a rectangle and a rhombus.

THEOREM XXXIV.

114. *Two parallelograms are equal when two adjacent sides and the included angle of one are equal respectively to two adjacent sides and the included angle of the other.*



In the parallelograms A B C D and A′ B′ C′ D′, let A B = A′ B′ and A D = A′ D′, and let the included angle A = angle A′.

To prove that the parallelograms are equal.

Since angle A = angle A′, the parallelogram A B C D may be superposed upon A′ B′ C′ D′ in such a way that the side A B shall coincide with its equal A′ B′, and A D with A′ D′.

Then since B C is parallel to A D and B′ C′ to A′ B′, the side B C will fall upon B′ C′, and the point C will fall somewhere in B′ C′.

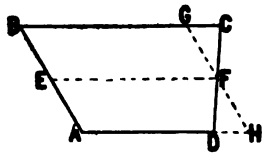
And since D C is parallel to A B and D′ C′ to A′ B′, the sides D C will fall upon D′ C′ and the point C will fall somewhere in D′ C′.

Therefore the point C, being at the same time in B′ C′ and D′ C′, must fall at their point of intersection C′.

Hence the parallelograms coincide throughout and are equal.

THEOREM XXXV.

115. *The line joining the middle points of the two non-parallel sides of a trapezoid is parallel to the two parallel sides, and equal to half their sum.*



Let E F join the middle points of the non-parallel sides A B and C D, of the trapezoid A B C D.

To prove:

I. That E F is parallel to A D and B C.

Through F draw G H parallel to B A, meeting A D produced, at H. The angles G F C and D F H are equal (vertical angles); also the angles G C F and F D H (alternate interior angles); and the side C F is equal to F D (hypothesis). Therefore the triangles G C F and F D H are equal (Theorem XVIII), and G F = F H = $\frac{1}{2}$ G H. But as A B G H is a parallelogram, G H = B A (Theorem XXIX); therefore F H = $\frac{1}{2}$ B A = A E.

Therefore A E F H is a parallelogram (Theorem XXX), and E F is parallel to A D, and therefore also to B C.

II. That E F = $\frac{1}{2}$ (A D + B C.)

Since A E F H and E B G F are parallelograms,

$$E F = A H = A D + D H$$

and also $$E F = B G = B C - G C$$

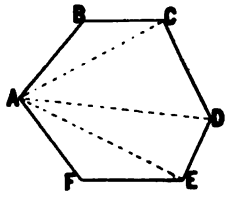
Now, as the two triangles G F C and D F H are equal, G C = D H; therefore if we add the two equations:

$$2 E F = A D + B C$$

$$E F = \tfrac{1}{2} (A D + B C)$$

THEOREM XXXVI.

116. *The sum of the interior angles of a polygon is equal to twice as many right angles as it has sides minus two.*



Let A B C D E F oe the given polygon.

To prove that the sum of the interior angles A, B, C, D, E, and F, is equal to twice as many right angles as the figure has sides minus two.

If from any vertex as A, diagonals A C, A D, A E, are drawn, the polygon will be divided into as many triangles as the figure has sides minus two; and since we know that the sum of the angles of each triangle is equal to two right angles, the sum of the angles of all the triangles, that is, the sum of the interior angles of the polygon, is equal to twice as many right angles as the polygon has sides minus two.

RATIO AND PROPORTION.

DEFINITIONS.

(NOTE. It is necessary to understand the elementary principles of ratio and proportion before entering upon the theorems that are to follow. It is therefore introduced here.)

117. Ratio is the relation of one quantity to another of the same kind; or it is the quotient obtained by dividing one quantity by another of the same kind.

Ratio is indicated by writing the two quantities thus:

$a : b$, or $\frac{a}{b}$; read a is to b, or a divided by b.

118. The **Terms** of a ratio are the quantities compared, whether simple or compound.

The first term of a ratio is called the *antecedent*, the other the *consequent*; the two terms together are called a *couplet*.

119. An **Inverse** or **Reciprocal Ratio** of any two quantities is the ratio of their *reciprocals*. Thus the *direct* ratio of a to b is $a : b$, that is $\frac{a}{b}$; the *inverse* ratio of a to b is $\frac{1}{a} : \frac{1}{b}$, that is $\frac{1}{a} \div \frac{1}{b} = \frac{b}{a}$, or $b : a$.

120. Proportion is an equality of ratios.

Four quantities are in proportion when the ratio of the first to the second equals the ratio of the third to the fourth.

The equality of two ratios is indicated by the sign of equality ($=$), or by four dots (: :).

Thus $a : b = c : d$, or $a : b : : c : d$, or $\frac{a}{b} = \frac{c}{d}$; read a to b equals c to d, a is to b as c is to d, or a divided by b equals c divided by d.

121. In a proportion, the antecedents and consequents of the two ratios are respectively the *antecedents* and *consequents* of the proposition. The first and fourth terms are called the *extremes*, the second and third the *means*.

122. In a proportion in which the means are equal, e. g., $a : b = b : c$, either mean is called a *mean proportional* between the other two; and the third, a *third proportional* to the first and second.

123. A proportion is transformed by **Alternation** when antecedent is compared with antecedent, and consequent with consequent.

124. A proportion is transformed by **Inversion** when the antecedents are made consequents and the consequents antecedents.

125. A proportion is transformed by **Composition** when in each couplet the sum of the antecedent and consequent is compared with the antecedent or with the consequent.

126. A proportion is transformed by **Division** when in each couplet the difference of the antecedent and consequent is compared with the antecedent or with the consequent.

127. **Axiom.** Two ratios respectively equal to a third are equal to each other.

The following are the theorems of ratio and proportion:

THEOREM I.

128. *In a proportion the product of the extremes is equal to the product of the means.*

Let $a : b = c : d$

That is $\frac{a}{d} = \frac{c}{b}$

Clearing of fractions, $ad = bc$

129. *Scholium.* A proportion is merely an equation; and when we make the product of the extremes equal to the product of the means, we simply clear of fractions.

THEOREM II.

130. *If (conversely to Theorem I) the product of the two quantities be equal to the product of two others, one pair may be made the extremes, and another the means of a proportion.*

Let $ad = bc$

Dividing by bd, $\frac{ad}{bd} = \frac{bc}{bd}$ or $\frac{a}{b} = \frac{c}{d}$.

That is, $a : b = c : d$

Similarly we may prove that:

$a : c = b : d$

$b : d = a : c$

$c : d = a : b$, etc.

THEOREM III.

131. *In any proportion the terms are in proportion by* **Alternation**; *that is, the first term is to the third as the second is to the fourth.*

Let $a:b = c:d$

By (128) $ad = bc$

By (130) $a:c = b:d$

THEOREM IV.

132. *In any proportion the terms are in proportion by* **Inversion**; *that is, the second term is to the first as the fourth is to the third.*

Let $a:b = c:d$

By (128) $ad = bc$

By (130) $b:a = d:c$

THEOREM V.

133. *In any proportion the terms are in proportion by* **Composition**; *that is, the sum of the first two terms is to the first term, as the sum of the last two terms is to the third term.*

Let $a:b = c:d$

That is $\frac{a}{b} = \frac{c}{d}$

Adding 1 to each member,

$$\frac{a}{b} + 1 = \frac{c}{d} + 1$$

or, $$\frac{a+b}{b} = \frac{c+d}{d}$$

That is, $a + b:b = c + d:d$

THEOREM VI.

134. *In any proportion the terms are in proportion by* **Division**; *that is, the difference of the first two terms is to the first term, as the difference of the last two terms is to the third term.*

Let $a:b = c:d$

That is, $\frac{a}{b} = \frac{c}{d}$

Subtracting 1 from each member, $\frac{a}{b} - 1 = \frac{c}{d} - 1$

or, $\frac{a-b}{b} = \frac{c-d}{d}$

That is, $a - b : b = c - d : d$

135. *Corollary.* From (133) and (134), and by means of (131) and (127):

If $a : b = c : d$

Then $a + b : a - b = c + d : c - d$

THEOREM VII.

136. ***Equimultiples of two quantities have the same ratio as the quantities themselves.***

For $\frac{a}{b} = \frac{ma}{mb}$

That is $a : b = ma : mb$

137. *Corollary.* It follows that either couplet of a proposition may be multiplied or divided by any quantity, and the resulting quantities will be in proportion. And since by (131), if $a : b = ma : mb$, $a : ma = b : mb$ or $ma : a = mb : b$, it follows that both consequents, or both antecedents, may be multiplied or divided by any quantity, and the resulting quantities will be in proportion.

THEOREM VIII.

138. ***In any proportion, like powers or like roots of the terms are in proportion.***

Let $a : b = c : d$

That is $\frac{a}{b} = \frac{c}{d}$

Hence, $\frac{a^n}{b^n} = \frac{c^n}{d^n}$

That is, $a^n : b^n = c^n : d^n$

Since n may be either practical or integral, the theorem is proved.

THEOREM IX.

139. *If any number of quantities are proportional, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.*

Let $a:b = c:d = e:f$

Now, $ab = ab$ (A)

And by (128), $ad = bc$ (B)

And also, $af = be$ (C)

Adding (A), (B), (C),

$$a(b + d + f) = b(a + c + e)$$

Hence by (130), $a:b = a + c + e : b + d + f$

THEOREM X.

140. If there are two sets of quantities in proportion, their products, or quotients, term by term, will be in proportion.

Let $a:b = c:d$

And $e:f = g:h$

By (128), $ad = bc$ (A)

And $eh = fg$ (B)

Multiplying (A) by (B), $adeh = bcfg$ (C)

Dividing (A) by (B), $\frac{ad}{eh} = \frac{bc}{fg}$ (D)

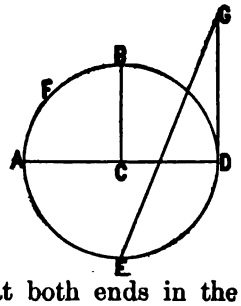
From (C) by (130), $ae:bf = cg:dh$

And from (D), $\frac{a}{e}:\frac{b}{f} + \frac{c}{g}:\frac{d}{h}$

THE CIRCLE.

DEFINITIONS.

141. A **Circle** is a plane figure bounded by a curved line called the *circumference*, every point of which is equally distant from a point within called the *centre*; as A B D E.



142. A **Radius** of a circle is a straight line joining the centre with the circumference, as B C.

143. A **Diameter** of a circle is a straight line drawn through the centre, terminating at both ends in the circumference, as A D.

144. *Corollary.* The radii of a circle, or of equal circles, are evidently equal; also the diameters are equal, and each diameter is equal to twice the radius.

145. An **Arc** is any part of the circumference; as A F B. When an arc equals one-half the circumference, it is called a *Semi-circumference;* when it equals one-fourth a circumference, it is called a *quadrant.*

146. A **Chord** is the straight line joining the ends of an arc, as for example, a straight line joining A and B. The arc is said to be *subtended* by its chord.

147. A **Segment** of a circle is the portion included between an arc and its chord.

148. A **Sector** is the part of a circle included between two radii and the intercepted arc; as the space B C D.

149. A **Tangent** (in geometry) is a straight line which touches, but does not, though produced, cut the circumference, as G D. The point at which the tangent touches the circumference is called the *point of contact,* or *point of tangency.*

150. A **Secant** is a straight line which, if sufficiently produced, will cut the circumference in two points, as G E.

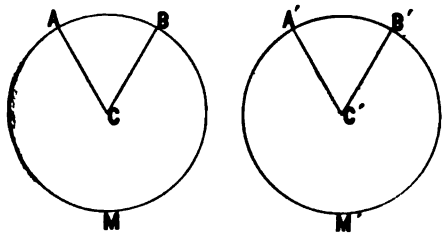
151. Two circles are *concentric* when they have the same centre.

152. A *central* angle is one whose vertex is at the centre, the sides being radii, as angle A C B.

An *inscribed* angle is one whose vertex is on the circumference, the sides being chords.

THEOREM XXXVII.

153. *In equal circles, or in the same circle, equal central angles intercept equal arcs.*



In the equal circles A M B and A′ M′ B′, let C and C′ be equal central angles.

To prove that arc A B = arc A′ B′:

Superpose the sector A B C upon A′ B′ C′ in such a way that the angle C shall coincide with its equal C′.

Then since the radii of the circles are equal, the point A will fall at A′ and B at B′.

Therefore the arc A B will coincide with the arc A′ B′, since all points of each are equally distant from the centre.

Hence, arc A B = arc A′ B′.

THEOREM XXXVIII.

154. Conversely, *in equal circles, or in the same circle, equal arcs are intercepted by equal central angles.*

In the equal circles A M B and A′ M′ B′, let the arc A B = arc A′ B′.

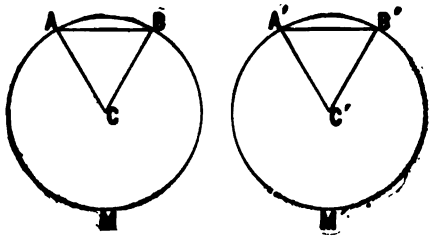
To prove that angle C = angle C′:

Since arc A B = arc A′ B′, the circle A B M may be superposed upon A′ B′ M′ in such a way that the point A shall fall at A′ and B at B′, the centre C falling at C′.

Then the radii A C and B C will coincide with A′ C′ and B′ C′ respectively.

THEOREM XXXIX.

155. *In equal circles, or in the same circle, equal chords subtend equal arcs.*



In the equal circles A M B and A′ M′ B′, let the chord A B be equal to chord A′ B′.

To prove that arc A B = arc A′ B′:

Let C and C′ be the centres of the circles, and draw A C, B C, A′ C′, and B′ C′.

Then in the triangles A B C and A′ B′ C′, A B = A′ B′, by hypothesis.

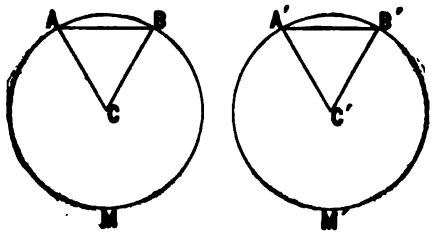
And since the radii of circles are equal, A C = A′ C′, and B C = B′ C′.

Hence, the triangles are equal, (Theorem XIX), and angle C = angle C′.

Hence, arc A B = arc A′ B′ (Theorem XXXVII).

THEOREM XL.

156. Conversely, *in equal circles, or in the same circle, equal arcs are subtended by equal chords.*



In the equal circles A M B and A′ M′ B′, let the arc A B = arc A′ B′.

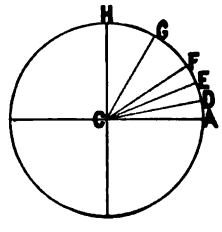
To prove that chord A B = chord A′ B′:

Let C and C′ be the centres of the circles; join A C, B C, A′ C′, and B′ C′.

Then in the triangles A B C and A′ B′ C′, A C = A′ C′, and B C = B′ C′. And since arc A B = arc A′ B′, angle C = angle C′, (Theorem XXXVIII), and chord A B = chord A′ B′.

THEOREM XLI.

157. *Central angles vary as their corresponding arcs.*



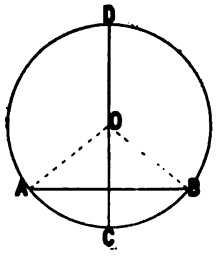
Let A C D, D C E, E C F be equal central angles at C; then the arcs A D, D E, E F are equal (Theorem XXXVII); then the angle A C E is double the angle A C D, and the arc A E is double the arc A D; and the angle A C F three times the angle A C D, and the arc A F three times the arc A D; and if the angle A C G is m times the angle A C D, the arc A G is m times the arc A D; that is, the angle varies as the arc, or the arc as the angle.

158. *Corollary.* As central angles vary as their arcs, or arcs vary as their corresponding angles, either of these quantities

may be assumed as the measure of the other. The measure of an angle is, then, *the arc included between its sides and described from its vertex as a center.*

THEOREM XLII.

159. *The diameter perpendicular to a chord bisects the chord and its subtended arc.*



In the circle A B D, let the diameter C D be perpendicular to the chord A B.

To prove that it bisects A B and its subtended arcs:

Let O be the centre of the circle; join O A, and O B.

Then since O A = O B, the triangle O A B is isosceles; and the line C D, passing through the vertex perpendicular to the base, bisects the base and also the vertical angle, (by Theorem XXI, Corollary I).

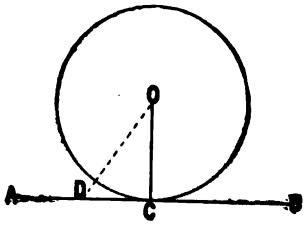
Hence, angle A O C = angle B O C, and arc A C = arc B C, (Theorem XXXVII).

Subtracting the equal arcs A C and B C from the semi-circumferences C A D and C B D, we have arc A D = arc B D.

Therefore the diameter bisects the chord A B and its subtended arcs A C B and A D B.

THEOREM XLIII.

160. *A straight line perpendicular to a radius at its extremity, is tangent to the circumference.*



Let A B be perpendicular to the radius O C, at its extremity C.

To prove that it is tangent to the circumference:

Since the perpendicular O C is the shortest line which can be drawn from O to A B (Theorem VIII), any other line, as O D, is greater than O C, and hence the point D lies without the circle.

That is, every point of A B except C lies without the circle, and A B is therefore tangent to the circumference (article 149).

161. *Corollary.* Conversely, *a tangent to the circumference is perpendicular to the radius drawn to the point of contact.*

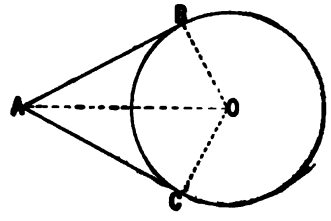
For if A B is tangent to the circumference at C, every point of A B except C lies without the circle.

Hence, O C is the shortest line which can be drawn from O to A B, and is therefore perpendicular to A B.

THEOREM XLIV.

162. *The two tangents to a circumference from an outside point are equal.*

Let A B and A C be the tangents from the point A to the circumference whose centre is O.



To prove that A B = A C.

Join O A, and draw the radii O B and O C. Since O B is perpendicular to A B, and O C to A C, (by Theorem XLIII, Corollary I), A B O and A C O are right angles.

Then in right triangles O A B and O A C, O A is common, and O B = O C.

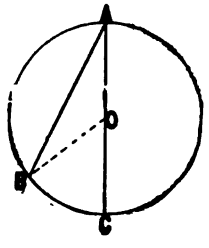
Hence the triangles are equal (by Theorem XX), and A B = A C

THEOREM XVI.

163. *An inscribed angle is measured by one-half its intercepted arc.*

Case I. *When one side of the angle is a diameter.*

Let A C be a diameter of the circle A B C.



To prove that the inscribed angle A is measured by one-half arc B C.

Draw the radius O B. Then (by Theorem XVI, Corollary VI), angle B O C = angle A + angle B.

But since A O B is an isosceles triangle (O A and O B being radii of circle A B C) angle A = angle B.

Hence, angle B O C = 2 angle A, or angle A = one-half angle B O C.

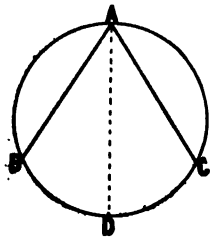
Now angle B O C is measured by arc B C (Theorem XLI, Corollary I.)

Therefore angle A is measured by one-half arc B C.

Case II. *When the centre is within the angle.*

In the circle A B C let the centre lie within the angle B A C.

To prove that angle B A C is measured by one-half arc B C.



Draw the diameter A D. Then angle B A C = B A D + C A D.

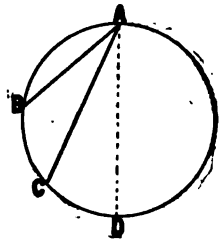
But by Case I, angle B A D is measured by one-half arc B D, and angle C A D is measured by one-half arc C D.

Hence angle B A C is measured by one-half (arc B D + arc C D), or by one-half arc B C.

Case III. *When the centre lies without the angles.*

In the circle A B C let the centre lie without the angle B A C.

To prove that angle B A C is measured by one-half arc B C.

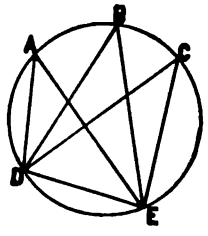


Draw the diameter A D. Then angle B A C = angle B A D — angle C A D.

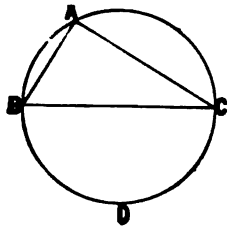
But by Case I, angle B A D is measured by one-half arc B D, and angle C A D is measured by one-half arc C D.

Hence angle B A C is measured by one-half (arc B D — arc C D), or by one-half arc B C.

164. *Corollary I. All the angles A, B, C, inscribed in the same segment are equal.*



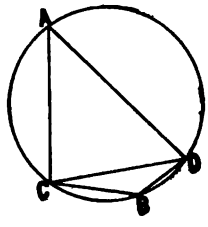
For each is measured by one-half arc D E.



165. *Corollary II. An angle B A C inscribed in a semi-circle is a right angle.*

For it is measured by one-half the semi-circumference B D C; that is, by a quadrant (Article 145.)

166. *Corollary III. An angle as C A D, inscribed in a segment greater than a semi-circle is acute.*



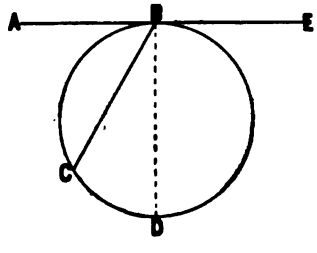
For it is measured by one-half an arc less than a semi-circumference, that is, by an arc less than a quadrant

An angle C B D inscribed in a segment less than a semi-circle is obtuse.

For it is measured by one-half an arc greater than a semi-circumference; that is by an arc greater than a quadrant.

THEOREM XLVI.

167. *An angle formed by a tangent and a chord is measured by one-half the intercepted arc.*



Let A E be tangent to the circumference B C D at B, and let B C be a chord.

To prove that angle A B C is measured by one-half arc B C.

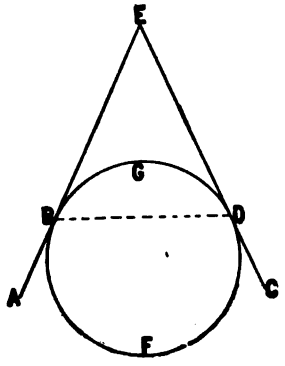
Draw the diameter B D perpendicular to A E (Theorem XLIII, Corollary I.) Then angle A B C = angle A B D — angle C B D. But since a right angle is measured by one-half a semi-circumference, angle A B D is measured by one-half arc B C D. And (by Theorem XLV), angle C B D is measured by one-half arc C D.

Therefore angle A B C is measured by one-half (arc B C D – arc C D), or by one-half arc B C.

Similarly, angle C B E is measured by one-half arc C D B.

THEOREM XLVII.

168. *The angle between two tangents is measured by one-half the difference of the intercepted arcs.*



Let A E and C E be tangent to the circumference B D F at B and D respectively.

To prove that angle E is measured by one-half (arc B F D — arc B G D.)

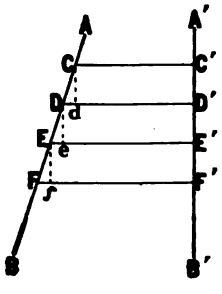
Draw the chord B D. Then the exterior angle A B D is equal to the sum of the opposite interior angles E and B D E. That is, angle E = angle A B D — angle B D E.

But (by Theorem XLVI), angle A B D is measured by one-half arc B F D, and angle F D E is measured by one-half arc B G D.

Therefore angle E is measured by one-half arc B F D — arc B G D).

THEOREM XLVIII.

169. *If a series of parallels, cutting two straight lines, intercept equal distances on one of these lines, they also intercept equal distances on the other line.*



Let the lines A B and A′ B′ be cut by the series of parallels C C′, D D′, E E′, and F F′; and let C D = D E = E F.

To prove that C′ D′ = D′ E′ = E′ F′:

Draw Cd, De, and Ef, parallel to A′ B′.

Then in the triangles C Dd, D Ee, and E Ff, by hypothesis, C D = D E = E F.

Also since the parallels D D′, E E′, F F′, are cut by A B, angle C Dd = angle D Ee = angle E Ff, (Theorem V).

And since the parallels Cd, De, and Ef are cut by A B, angle D Cd = angle E De = angle F Ef (Theorem V).

Therefore the triangles are equal, (Theorem XVIII), and Cd = De = Ef.

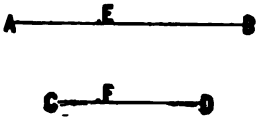
But in the parallelograms C C′ D′d, D D′ E′e, and E E′ F′f, we have

Cd = C′ D′, De = D′ E′, and Ef = E′ F′ (Theorem XXIX).

Therefore, C′ D′ = D′ E′ = E′ F′.

170. Definition. Two straight lines are said to be divided *proportionally* when their corresponding segments are in the same ratio as the lines themselves.

Thus the lines A B and C D are divided proportionally at E and F if A B : A E = C D : C F.

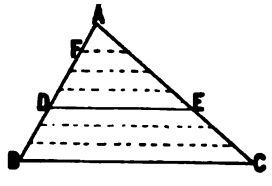


THEOREM XLIX.

171. ***A line parallel to one side of a triangle divides the other two sides proportionally.***

Let D E be parallel to side B C of triangle A B C.

To prove that $\frac{A B}{A D} = \frac{A C}{A E}$



When A B *and* A D *are commensurable, i. e., have a common unit of measure.*

Let A F be a common measure of A B and A D, and suppose it to be contained seven times in A B and four times in A D.

$$\text{Then } \frac{A B}{A D} = \frac{7}{4}. \quad (1)$$

Drawing parallels to B C through the several points of division, the side A C will be divided into seven equal parts (Theorem XLVIII), of which A E will contain four.

$$\text{Then } \frac{A C}{A E} = \frac{7}{4}. \quad (2)$$

From (1) and (2), by axiom,

$$\frac{A B}{A D} = \frac{A C}{A E}.$$

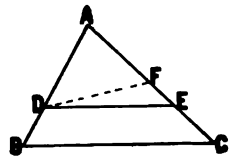
When A B *and* A D *are incommensurable.*

This may be proved also, but depends on considerations not undertaken in this book:

THEOREM L.

172. Conversely, *a straight line which divides two sides of a triangle proportionally is parallel to the third side.*

In the triangle A B C, let D E be drawn so that $\frac{A B}{A D} = \frac{A C}{A E}$.



To prove that D E is parallel to B C:

If D E is not parallel to B C, let D F be drawn parallel to B C.

Then by (Theorem XLIX) $\frac{A B}{A D} = \frac{A C}{A F}$.

But by hypothesis, $\frac{A B}{A D} = \frac{A C}{A E}$.

Whence $\frac{A C}{A E} = \frac{A C}{A F}$.

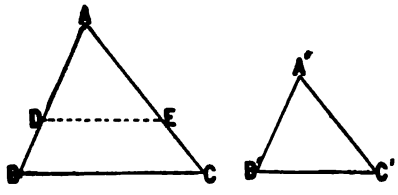
Therefore, A E = A F, which is impossible unless D F coincides with D E.

Hence, D E is parallel to B C.

173. Definition. Two polygons are called *similar* when they are mutually equiangular, and have their homologous sides proportional.

THEOREM LI.

174. *Two triangles are similar when they are mutually equiangular.*



In the triangles A B C and A′ B′ C′, let angle A = angle A′, angle B = angle B′, and angle C = angle C′.

To prove that the triangles are similar:

Superpose the triangle A′ B′ C′ upon A B C so that angle A′ shall coincide with its equal angle A, the side B′ C′ falling at D E.

Then since angle A D E = angle B, D E is parallel to B C (Theorem VI).

Hence, (by Theorem XXIX),

$$\frac{AB}{AD} = \frac{AC}{AE}, \text{ or } \frac{AB}{A'B'} = \frac{AC}{A'C'}. \quad (1)$$

Similarly, by superposing the triangle A′ B′ C′ upon A B C so that angle B′ shall coincide with angle B, it may be proved that

$$\frac{AB}{A'B'} = \frac{BC}{B'C'}. \quad (2)$$

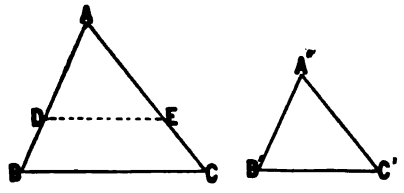
From (1) and (2) we have

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}.$$

Therefore, the homologous sides are proportional and the triangles are similar.

THEOREM LII.

175. *Two triangles are similar when their homologous sides are proportional.*



In the triangles A B C and A′ B′ C′, let

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$$

To prove that the triangles are similar.

Take A D = A′ B′, and A E = A′ C′, and join D E. Then from the given proportion we have

$$\frac{AB}{AD} = \frac{AC}{AE}$$

Therefore the line D E is parallel to B C (Theorem L.)

Then since the parallels D E and B C are cut by A B, angle A D E = angle B, and angle A E D = angle C.

Hence the triangles A B C and A D E are mutually equiangular, and therefore similar (Theorem LI.)

Then their homologous sides are proportional; that is,

$$\frac{AB}{AD} = \frac{BC}{DE}, \text{ or } \frac{AB}{A'B'} = \frac{BC}{D'E'}$$

But by hypothesis, $\frac{AB}{A'B'} = \frac{BC}{B'C'}$

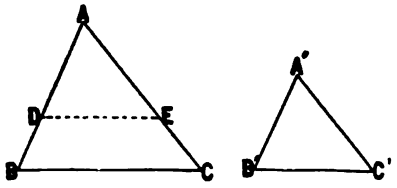
Hence D E = B′ C′, and the triangles A D E and A′ B′ C′ are mutually equilateral, and therefore equal (Theorem XIX.)

But the triangle A D E has been proved similar to A B C.

Hence the triangle A′ B′ C′ is similar to A B C.

THEOREM LIII.

176. *Two triangles are similar when they have an angle of one equal to an angle of the other, and the sides including these angles proportional.*



In the triangles A B C and A′ B′ C′, let angle A = angle A and $\frac{A\,B}{A'\,B'} = \frac{A\,C}{A'\,C'}$

To prove that the triangles are similar.

Superpose the triangle A′ B′ C′ upon A B C in such a way that the angle A′ shall coincide with its equal A, the side B′ C′ falling at D E.

Then by hypothesis, $\frac{A\,B}{A\,D} = \frac{A\,C}{A\,E}$

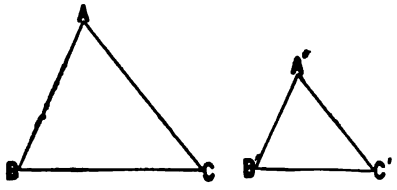
Therefore the side D E is parallel to B C (Theorem L), and angle A D E = angle B and angle A E D = angle C.

Hence the triangles A B C and A′ B′ C′ are mutually equiangular, and therefore similar.

That is, the triangles A B C and A′ B′ C′ are similar.

THEOREM LIV.

177. *Two triangles are similar when the sides of one are parallel or perpendicular respectively to the sides of the other.*



In the triangles A B C and A′ B′ C′, let A′ B′ be parallel or perpendicular to A B, A′ C′ to A C, and B′ C′ to B C.

To prove that the triangles are similar.

Since the sides of the angles A and A′ are parallel or perpendicular each to each, the angles are either equal or supplementary, for they have the same differences of direction (Theorem IV); and the same is true of the angles B and B′, C and C′.

We may then make three hypotheses with regard to the angles of the triangles, R denoting a right angle :—

FIRST. If supplementary.	SECOND. If one pair is equal and two supplementary.	THIRD. If equal.
$A + A' = 2\,R$	$A = A'$	$A = A'$
$B + B' = 2\,R$	$B + B' = 2\,R$	$B = B'$
$C + C' = 2\,R$	$C + C' = 2\,R$	$\therefore C = C'$

(Theorem XVI, Corollary II.)

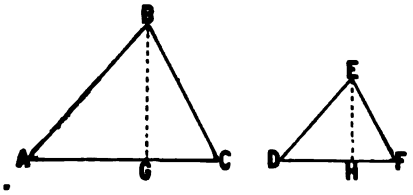
But the first and second hypotheses are impossible, since, in either case, the sum of the angles of the two triangles would exceed four right angles.

We can then only have $A = A'$, $B = B'$, and $C = C'$.

Hence the triangulars are mutually equiangular, and therefore similar. (Theorem LI.)

THEOREM LV.

178. *The homologous altitudes of two similar triangles are proportional to the homologous sides.*



In the similar triangles A B C and D E F, let B G and E H be homologous altitudes

To prove that

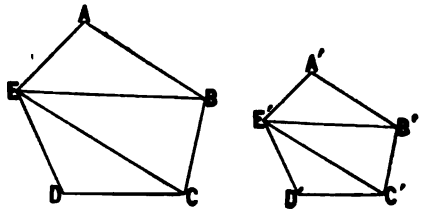
$$\frac{BG}{EH} = \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

The two right triangles A B G and D E H are mutually equiangular (Theorem XVI, Corollary II), and hence similar (Theorem LI.)

Therefore, $\frac{BG}{EH} = \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$

THEOREM LVI.

179. *Two polygons are similar when they are composed of the same number of triangles, similar each to each and similarly placed.*



In the polygons A — E and A′— E′, the triangle A B E will be similar to A′ B′ E′, B C E to B′ C′ E′, and C D E to C′ D′ E′.

To prove that the polygons are similar.

I. The polygons are mutually equiangular.

For since two similar triangles are mutually equiangular,

$$\text{angle A} = \text{angle A}'.$$

Also angle A B E + angle E B C = angle A′ B′ E′ + angle E′ B′ C′.

Or, $$\text{angle A B C} = \text{A}' \text{B}' \text{C}'.$$

In like manner, $$\text{angle B C D} = \text{angle B}' \text{C}' \text{D}'.$$

And $$\text{angle D} = \text{angle D}'.$$

II. The polygons have their homologous sides proportional.

For since two similar triangles have their homologous sides proportional,

$$\frac{\text{A B}}{\text{A}' \text{B}'} = \frac{\text{B E}}{\text{B}' \text{E}'}, \text{ and } \frac{\text{B E}}{\text{B}' \text{E}'} = \frac{\text{B C}}{\text{B}' \text{C}'}$$

Whence, $$\frac{\text{A B}}{\text{A}' \text{B}'} = \frac{\text{B C}}{\text{B}' \text{C}'}$$

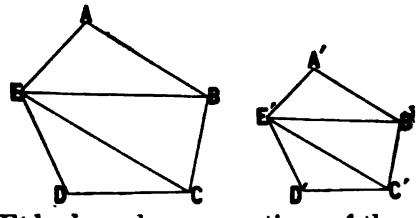
In like manner we may prove,

$$\frac{\text{A B}}{\text{A}' \text{B}'} = \frac{\text{B C}}{\text{B}' \text{C}'} = \frac{\text{C D}}{\text{C}' \text{D}'}, \text{ etc.}$$

Therefore the polygons are similar, (article 173.)

THEOREM LVII.

180. Conversely, *two similar polygons may be decomposed into the same number of triangles, similar each to each and similarly placed.*



Let E and E′ be homologous vertices of the similar polygons A — E and A′ — E′, and draw the diagonals E B, E C, E′ B′, and E′ C′.

To prove that the triangle A B E is similar to A′ B′ E′, B C E to B′ C′ E′, and C D E to C′ D′ E′.

Since the polygons are similar (Article 173), angle A = angle A′, and $\frac{A\,E}{A'\,E'} = \frac{A\,B}{A'\,B'}$

Therefore the triangles A B E and A′ B′ E′ are similar, for they have an angle of one equal to an angle of the other, and the sides including these angles proportional (Theorem LIII).

Again, since the polygons are similar, angle A B C = angle A′ B′ C′.

And since the triangles A B E and A′ B′ E′ are similar, angle A B E = angle A′ B′ E′.

Subtracting these equals, we have angle A B C — angle A B E = angle A′ B′ C′ — angle A′ B′ E′, or angle E B C = angle E′ B′ C′.

Also, from the similarity of the polygons,

$$\frac{A\,B}{A'\,B'} = \frac{B\,C}{B'\,C'}$$

And since the triangles A B E and A′ B′ E′ are similar,

$$\frac{A\,B}{A'\,B'} = \frac{B\,E}{B'\,E'}$$

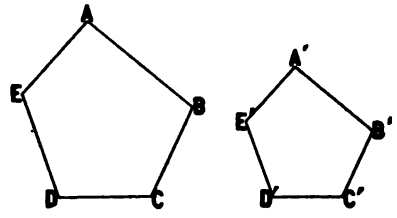
Whence, $\frac{B\,C}{B'\,C'} = \frac{B\,E}{B'\,E'}$

Therefore the triangles B C E and B′ C′ E′ are similar (Theorem LIII).

In like manner it may be proved that the triangles C D E and C′ D′ E′ are similar.

THEOREM LVIII.

181. *The perimeters of two similar polygons are in the same ratio as any two homologous sides.*



Let P and P′ denote the perimeters of the similar polygons A — E, and A′ — E′.

To prove that $\frac{P}{P'} = \frac{A\,B}{A'\,B'} = \frac{B\,C}{B'\,C'} = \frac{C\,D}{C'\,D'}$, etc.

Since the polygons are similar, we have,

$$\frac{A\,B}{A'\,B'} = \frac{B\,C}{B'\,C'} = \frac{C\,D}{C'\,D'}, \text{ etc.}$$

Whence by Article 139,

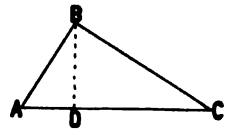
$$\frac{A\,B + B\,C + C\,D + \text{etc.}}{A'\,B' + B'\,C' + C'\,D' + \text{etc.}} = \frac{A\,B}{A'\,B'} = \frac{B\,C}{B'\,C'} = \frac{C\,D}{C'\,D'}, \text{ etc.}$$

That is $\frac{P}{P'} = \frac{A\,B}{A'\,B'} = \frac{B\,C}{B'\,C'} = \frac{C\,D}{C'\,D'}$, etc.

THEOREM LIX.

182. *In a right triangle the perpendicular drawn from the vertex of the right angle to the hypothenuse divides the triangle into two triangles similar to the whole triangle and to each other.*

In the right triangle A B C, let B D be drawn from the vertex B of the right angle perpendicular to the hypothenuse A C.



To prove that the two triangles A B D and B D C are similar to the triangle A B C and to each other.

The two right triangles A B D and A B C have the acute angle A common; they are therefore mutually equiangular (Theorem XVI, Corollary II), and similar (Theorem LI).

The two right triangles A B C and B C D have the acute angle C common; therefore, they are mutually equiangular and similar.

The two triangles A B D and B C D, being similar each to A B C, are similar to each other.

183. *Corollary I.* Since A B C and A B D are similar triangles,

$$\frac{\overline{A\,C}}{\overline{A\,B}} = \frac{\overline{A\,B}}{\overline{A\,D}}$$

And since A B C and B C D are similar,

$$\frac{\overline{A\,C}}{\overline{C\,B}} = \frac{\overline{C\,B}}{\overline{C\,D}},$$

that is, *if in a right triangle, a perpendicular is drawn from the vertex of the right angle to the hypothenuse, either side about the right angle is a mean proportional between the whole hypothenuse, and the adjacent segment.*

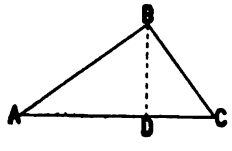
184. *Corollary II.* As A B D and B C D are similar triangles,

$$\frac{\overline{A\,D}}{\overline{D\,B}} = \frac{\overline{D\,B}}{\overline{D\,C}},$$

that is, *in a right triangle the perpendicular from the vertex of the right angle is a mean proportional between the segments of the hypothenuse.*

THEOREM LX.

185. *The square described on the hypothenuse of a right triangle is equivalent to the sum of the squares described upon the other sides.*



Let A B C be a right triangle.

To prove that $\overline{A\,B}^2 + \overline{B\,C}^2 = \overline{A\,C}^2$.

Draw B D perpendicular to A C.

Now (by Theorem LIX, Corollary I),

$$A\,C : A\,B = A\,B : A\,D.$$

That is, $\overline{A\,B}^2 = A\,C \times A\,D.$

And in the same way,

$$\overline{B\,C}^2 = A\,C \times C\,D.$$

Adding this equations,

$$\overline{AB}^2 + \overline{BC}^2 = AC \times AD + AC \times CD.$$
$$= AC \times (AD + CD)$$
$$= AC \times AC$$
$$= \overline{AC}^2$$

Therefore, $\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$

186. Corollary I. Since

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2,$$
$$\overline{BC}^2 = \overline{AC}^2 - \overline{AB}^2$$

And $$BC = \sqrt{AC^2 - AB^2}.$$

AREAS OF POLYGONS.

DEFINITIONS.

187. The **Area** of a polygon is the measure of its surface. It is expressed in units, which represent the number of times the polygon contains the square unit that is taken as a standard. This unit is called the *unit of surface*, and is usually adopted arbitrarily.

188. The usual unit of surface is the square whose side is some *unit of length ;* for example, a *square inch*, or a *square foot.*

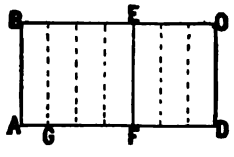
189. **Equivalent Polygons** are those which have the same area.

190. The *projection of a point* upon a straight line of indefinite length is the foot of the perpendicular drawn from the point to the line. An entire line may be said to have a *projection* upon another line in the same way.

191. A *medial line* of a triangle is a straight line drawn from either vertex to the middle point of the opposite side.

THEOREM LXI.

192. *Two rectangles having equal altitudes are to each other as their basis.*



Let A B C D and A B E F be two rectangles having equal altitudes.

To prove that $\frac{A\,B\,C\,D}{A\,B\,E\,F} = \frac{A\,D}{A\,F}$

Let A G be a common measure of A D and A F, and suppose it to be contained seven times in A D and four times in A F.

Then $$\frac{A\,D}{A\,F} = \frac{7}{4} \qquad (1)$$

Drawing perpendiculars to A D through the several points of division, the rectangle A B C D will evidently be divided into seven equal parts, of which the rectangle A B E F will contain four.

Then $$\frac{A\,B\,C\,D}{A\,B\,E\,F} = \frac{7}{4} \qquad (2)$$

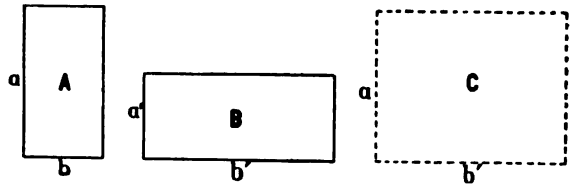
Then by (1) and (2) $\frac{A\,B\,C\,D}{A\,B\,E\,F} = \frac{A\,D}{A\,F}$, (things equal to the same thing are equal to each other, axiom.)

This may also be proved when A D and A F are incommensurable.

Corollary. Since either side of a rectangle may be taken as a base, it follows that *two rectangles with equal bases are to each other as their altitudes.*

THEOREM LXII.

193. *Any two rectangles are to each other as the products of their bases by their altitudes.*



Let A and B be any two rectangles leaving the altitudes a and a', and the bases b and b', respectively.

To prove that $\frac{A}{B} = \frac{a \times b}{a' \times b'}$

Let C be a third rectangle, having its altitude equal to the altitude of A and its base equal to the base of B.

Then the rectangles A and C, having equal altitudes, are to each other as their bases, (by Theorem LXI); that is,

$$\frac{A}{C} = \frac{b}{b'} \qquad (1)$$

And the rectangles C and B, having equal bases, are to each other as their altitudes (Theorem LXI, Corollary); that is,

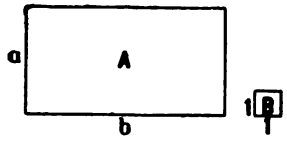
$$\frac{C}{B} = \frac{a}{a'} \qquad (2)$$

Multiplying equations (1) and (2), we have

$$\frac{A}{B} = \frac{a \times b}{a' \times b'}$$

THEOREM LXIII.

194. *The area of a rectangle is equal to the product of its base and altitude.*



Let a and b be the numerical measures of the altitude and base of the rectangle A, and let B be the square whose side is the unit of length.

To prove that the area of A, referred to B as a unit, is equal to a times b.

Since any two rectangles are to each other as the products of their bases by their altitudes (Theorem LXII), we have

$$\frac{A}{B} = \frac{a \times b}{1 \times 1} = a \times b$$

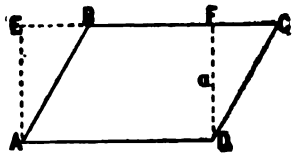
But since B is the unit of surface, $\frac{A}{B}$ is the area of A. (Article 186).

Hence, area of $A = a \times b$.

195. *Corollary.* The area of a square is equal to the square of one of its sides.

THEOREM LXIV.

196. *The area of a parallelogram is equal to the product of its base and altitude.*



Let A B C D be a parallelogram leaving its altitude D F equal to a and its base A D equal to b.

To prove that A B C D $= a \times b$:

Draw A E perpendicular to A D and meeting B C produced at E.

Then A E F D is a rectangle having its altitude equal to a and its base equal to b.

In the right triangles A B E and D C F, A B = D C and A E = D F (Theorem XXIX).

Therefore, the triangles are equal (Theorem XX). Now if from the entire figure A E C D, we take the triangle A B E, there remains the parallelogram A B C D; and if we take the triangle D C F, there remains the rectangle A E F D.

Therefore, area A B C D = area A E F D.

But the area of a rectangle A E F D $= a \times b$ (Theorem LXIII).

Hence, area A B C D $= a \times b$.

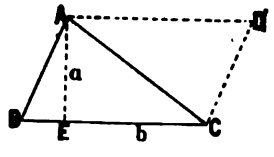
THEOREM LXV.

197. *The area of a triangle is equal to one-half the product of its base and altitude.*

Let A B C be a triangle, having its altitude equal to a and its base equal to b.

To prove that area A B C $= \frac{1}{2}\, a \times b$:

Draw the lines A D and C D parallel to B C and A B.



Then A B C D is a parallelogram having its altitude equal to a and its base equal to b.

Now a diagonal of a parallelogram divides it into equal triangles (Theorem XXIX).

Therefore area A B C $= \frac{1}{2}$ area A B C D.

But the area of the parallelogram A B C D is equal to a times b, (Theorem LXIV).

Hence, area A B C $= \frac{1}{2} a \times b$.

198. *Corollary I.* Two triangles having equal bases and equal altitudes are equivalent.

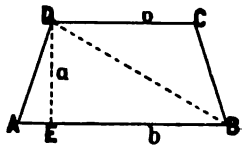
199. *Corollary II.* Two triangles having equal altitudes are to each other as their bases; two triangles having equal bases are to each other as their altitudes; and any two triangles are to each other as the products of their bases by their altitudes.

200. *Corollary III.* A triangle is equivalent to one-half a parallelogram having the same base and altitude.

THEOREM LXVI.

501. *The area of a trapezoid is equal to one-half the sum of its parallel sides multiplied by its altitude.*

Let A B C D be a trapezoid, having its altitude equal to a, and its parallel sides equal to b and b' respectively.



To prove that area A B C D $= a \times \frac{1}{2} (b + b')$:

Draw the diagonal B D dividing the trapezoid into two triangles A B D and B C D, having the common altitude a and their bases equal to b and b' respectively.

Then, area A B C D = area A B D + area B C D.

But since the area of a triangle is equal to one-half the product of its base by its altitude, (Theorem LXV),

area A B D $= \frac{1}{2} a \times b$,

and area B C D $= \frac{1}{2} a \times b'$.

Therefore, area A B C D $= \frac{1}{2} a \times b + \frac{1}{2} a \times b'$,

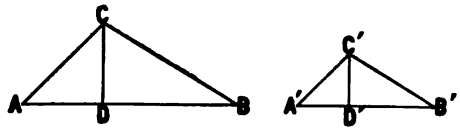
$= a \times \frac{1}{2} (b + b')$.

202. *Corollary.* Since the line joining the middle points of the non-parallel sides of a trapezoid is equal to one-half the sum of the bases (Theorem XXXV), it follows that,

The area of a trapezoid is equal to the product of its altitude by the line joining the middle points of the non-parallel sides.

THEOREM LXVII.

203. *Two similar triangles are to each other as the squares of their homologous sides.*



Let A B and A′ B′ be homologous sides of the similar tr. angles A B C and A′ B′ C′.

To prove that $$\frac{A\ B\ C}{A'\ B'\ C'} = \frac{\overline{A\ B}^2}{\overline{A'\ B'}^2}$$

Draw the altitudes C D and C′ D′.

Then since any two triangles are to each other as the products of their bases by their altitudes (Theorem LXV, Corollary II), we have

$$\frac{A\ B\ C}{A'\ B'\ C'} = \frac{A\ B \times C\ D}{A'\ B' \times C'\ D'} = \frac{A\ B}{A'\ B'} \times \frac{C\ D}{C'\ D'} \qquad (1)$$

But the homologous altitudes of two similar triangles are in in the same ratio as any two homologous sides (Theorem LV); hence,

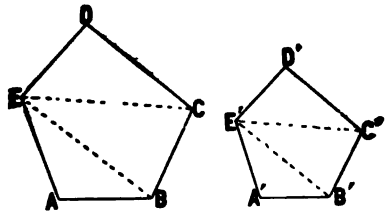
$$\frac{C\ D}{C'\ D'} = \frac{A\ B}{A'\ B'}$$

Substituting in (1), we have

$$\frac{A\ B\ C}{A'\ B'\ C'} = \frac{A\ B}{A'\ B'} \times \frac{A\ B}{A'\ B'} = \frac{\overline{A\ B}^2}{\overline{A'\ B'}^2}$$

THEOREM LXVIII.

204. *Two similar polygons are to each other as the squares of their homologous sides.*



Let K and K′ denote the areas of the similar polygons A — E and A′ — E′.

To prove that $$\frac{K}{K'} = \frac{\overline{A\ B}^2}{\overline{A'\ B'}^2}$$

Draw the diagonals E B, E C, E′ B′, and E′ C′.

Then since the triangle A B E is similar to A′ B′ E′, (Theorem LVII), we have

$$\frac{A\ B\ E}{A'\ B'\ E'} = \frac{\overline{A\ B}^2}{\overline{A'\ B'}^2} \text{ (Theorem LXVII).}$$

In like manner,

$$\frac{B\ C\ E}{B'\ C'\ E'} = \frac{\overline{B\ C}^2}{\overline{B'\ C'}^2} \text{ and } \frac{C\ D\ E}{C'\ D'\ E'} = \frac{\overline{C\ D}^2}{\overline{C'\ D'}^2}$$

But from the similarity of the given polygons,

$$\frac{\overline{A\ B}^2}{\overline{A'\ B'}^2} = \frac{\overline{B\ C}^2}{\overline{B'\ C'}^2} = \frac{\overline{C\ D}^2}{\overline{C'\ D'}^2}$$

Therefore,

$$\frac{A\ B\ E}{A'\ B'\ E'} = \frac{B\ C\ E}{B'\ C'\ E'} = \frac{C\ D\ E}{C'\ D'\ E'}$$

Whence,

$$\frac{A\ B\ E + B\ C\ E + C\ D\ E}{A'\ B'\ E' + B'\ C'\ E' + C'\ D'\ E'} = \frac{A\ B\ E}{A'\ B'\ E'} = \frac{\overline{A\ B}^2}{\overline{A'\ B'}^2}$$

That is,

$$\frac{K}{K'} = \frac{\overline{A\ B}^2}{\overline{A'\ B'}^2}$$

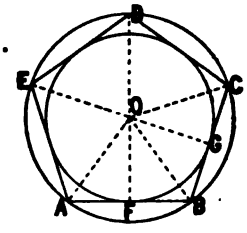
by ratio and proportion (Theorem IX).

REGULAR POLYGONS AND MEASUREMENT OF THE CIRCLE.

205. Definition. *A regular polygon is one which is both equilateral and equiangular.*

THEOREM LXIX.

206. *A circle may be circumscribed about, or inscribed within, any regular polygon.*



Let A B C D E be a regular polygon.

I. To prove that a circle may be circumscribed about it:

Let O be the centre of the circle passing through the vertices A, B, and C (the point O, at which perpendiculars erected at the middle points of A B and B C intersect, will be the centre of such

a circle, A O, B O, and C O, being equal, and therefore radii (by Theorem XI); join O A, O B, O C, and O D.

Then since the polygon is equiangular, angle A B C = angle B C D.

And since the triangle O B C is isosceles, angle O B C = angle O C B.

Subtracting these equals, we have angle O B A = angle O C D.

Also, O B = O C, being equal radii; and since the polygon is equilateral, A B = C D.

Therefore the triangles O A B and O C D are equal (Theorem XVIII) and O A = O D.

Hence the circle passing through A, B, and C also passes through D.

In like manner it may be shown that the circle passing through B, C, and D, also passes through E.

Therefore a circle may be circumscribed about A B C D E.

II. To prove that a circle may be inscribed in A B C D E.

Since the sides A B, B C, C D, etc., are equal chords of the circumscribed circle, they are equally distant from the centre.

For, drawing O F and O G perpendicular to A B and B C respectively, we have F as the middle point of A B, and G of B C, (by Theorem XLII), and the right triangle O F B and O G B are equal (by Theorem XX.)

Hence O F = O G and the circle described with O as a centre and O F as a radius will be inscribed in the polygon A D C D E.

207. Definitions. The *radius* of a regular polygon is the radius of its circumscribed circle, as O A.

The *apothem* is the radius of the inscribed circle, as O F.

The *centre* is the common centre O of the circumscribed and inscribed circles.

The *angle at the centre* is the angle, as A O B, between radii drawn to the extremities of any side.

208. *Corollary. Each angle at the centre of a regular polygon is equal to four right angles divided by the number of sides.*

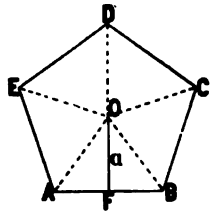
For from the equal triangles O A B, O B C, etc., we have, angle A O B = angle B O C = angle C O D, etc.

Therefore each angle is equal to four right angles divided by the number of sides (Theorem I.)

THEOREM LXX.

209. *The area of a regular polygon is equal to one-half the product of its perimeter and apothem.*

Let a denote the apothem O F, and P the perimeter of the regular polygon A B C D E.



To prove that area A B C D E = one-half P $\times$ a.

By drawing the radii O A, O B, O C, etc., the polygon may be divided into a series of triangles O A B, O B C, etc., whose common altitude is a.

Then (by Theorem LXV),

Area O A B = one-half A B $\times$ a.

Area O B C = one-half B C $\times$ a, and so on.

Adding, we have,

Area O A B + area O B C + etc., = one-half (A B + B C + etc.) $\times$ a.

That is, area A B C D E = one-half P $\times$ a.

The following important theorem is given more in the light of an example than a proof.

THEOREM LXII.

I. An inscribed or circumscribed polygon of an infinite number of sides practically coincides with the inscribing or circumscribing circle.*

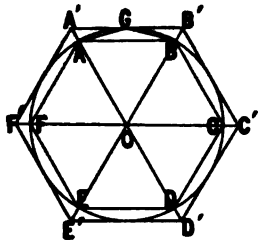
II. The areas of the above polygons are practically the same as that of the circle.

III. The approximate ratio of the circumference of a circle to its diameter is 3.1416.

IV. The area of the circle is 3.1416 *times the square of the radius.*

* To speak mathematically the perimeter of the inscribed or circumscribed polygon approaches the circumference of the circle and the areas of the polygons approach the area of the circle as a limit as the number of sides of each polygon is indefinitely increased. The demonstration here given will, however, suffice for the purposes of this book.

I. Let the number of sides of the inscribed and circumscribed polygons be doubled. It is readily seen that the sides of the polygons approach more closely the circumference, as is seen by replacing the side A B by the sides A G and B G of a polygon containing twice as many sides as the polygon A — F. If the number of sides were infinite, there would be no appreciable space between the circumference and the perimeter of the polygon. Hence they would practically coincide.



II. The area of the polygon A F is evidently equal to the sum of the triangles A O B, B O C, etc. Now if the number of sides of the polygon is increased indefinitely it is evident, as there is no appreciable area between the perimeter and circumference (I), that the sum of the areas of the infinite number of triangles thus formed is practically equal to the area of the circle.

III. By means of formulas, which we will not here attempt to derive, it is found that the perimeter of a polygon of 512 sides circumscribed about a circle whose diameter is 1, is 3.14163 while the perimeter of a 512 sided polygon inscribed within the above circle is 3.14157. The greater the number of sides to the polygons the nearer do these values approach each other. The value of the circumference must be between these values. The above figures are sufficient for our purpose. We, therefore, take as our approximate ratio of the circumference to the diameter 3.1416, i.e., denoting the circumference by C and the diameter by D we have $\frac{C}{D} = 3.1416$ or $C = 3.1416\ D$. The ratio is usually designated by the Greek letter Pi, π. Hence we say $C = \pi D$ or the circumference is π times the diameter. This holds for all circles.

IV. The area of a polygon is equal to one-half the value of the product of the perimeter and apothem. In an inscribed or circumscribed polygon, the apothem becomes practically equal to the radius of the circle when the number of sides of the polygon become infinite.

Hence the area of any circle is equal to one-half the circumference (πD) times the radius (which we will denote by r.) As the diameter is twice the radius, we have $D = 2r$. Hence the area of the circle is $\frac{1}{2} \times \pi\ 2r \times r = \pi r^2$. Writing $\frac{D}{2}$ in place of r we have $\frac{1}{4} \pi D^2$, the approximate value of π being 3.1416.

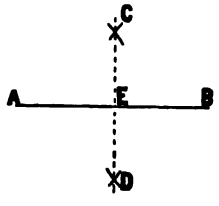
PROBLEMS OF CONSTRUCTION.

We have hitherto assumed that our figures were already constructed. The Problems of Construction here following depend for their solution upon principles already explained; and therefore, when the demonstration is not given in full, the theorems necessary to prove the construction will be referred to in the order in which they are to be used, and the student must complete the demonstration.

PROBLEM I.

1. *To bisect a given straight line.*

Let A B be the given straight line. From A and B as centres, with a radius greater than half of A B, describe arcs cutting one another at C and D; join C and D cutting A B at E, and the line A B is bisected at E.

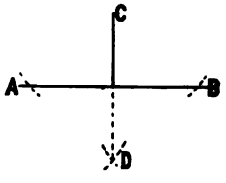


For C and D being each equally distant from A and B, the line C D must be perpendicular to A B at its middle point. (Converse of Theorem VIII.)

PROBLEM II.

2. *From a given point without a straight line to draw a perpendicular to that line.*

Let C be the point and A B the line. From C as a centre, describe an arc cutting A B in two points A and B; with A and B as centres, with a radius greater than half A B, describe arcs intersecting at D. Draw C D, and it is the perpendicular required. (Converse of Theorem VII.)

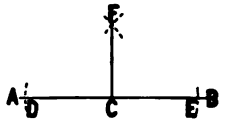


PROBLEM III.

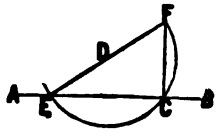
3. *From a given point in a straight line to erect a perpendicular to that line.*

Let C be the given point and A B the given line.

With C as a centre describe an arc cutting A B in D and E; with D and E as centres, with a radius greater than D C, describe arcs intersecting at F. Draw C F, and it is the perpendicular required. (Converse of Theorem VII.)



Second Method. With any point, D, without the line A B, with a radius equal to the distance from D to C, describe an arc cutting A B at E; draw the diameter E D F. Draw C F, and it is the perpendicular required. (Theorem XLV.)

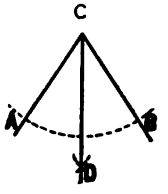


PROBLEM IV.

4. *To bisect a given arc, or angle.*

1st. Let A B be the given arc. Draw the chord A B and bisect it with a perpendicular.

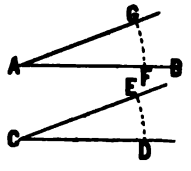
2nd. Let C be the given angle. With C as a centre, describe an arc cutting the sides of the angle in A and B; bisect the arc with the line C D, and it will also bisect the angle C. (Theorem XXXVII.)



PROBLEM V.

5. *At a given point in a straight line to make an angle equal to a given angle.*

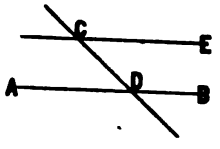
Let A be the given point in the line A B, and C the given angle. With C as a centre describe an arc D E cutting the sides of the angle C; with A as a centre, with the same radius, describe an arc F G; with F as a centre, with a radius equal to the distance from D to E, describe an arc cutting the arc F G. Draw A G. The angle A = angle C. (Theorem XXXVII and XXXIX.)



PROBLEM VI.

6. *Through a given point to draw a line parallel to a given straight line.*

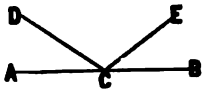
Let C be the given point, and A B the given line. From C draw a line C D to A B; at C in the line C D make an angle D C E equal to angle C D A (by Problem V;) C E is parallel to A B. (Theorem VI.)



PROBLEM VII.

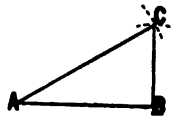
7. *Two angles of a triangle given to find the third.*

Draw an indefinite line A B; at any point C make an angle A C D equal to one of the given angles, and angle D C E equal to the other (by Problem V.) Then E C B is equal to the third angle. (Theorems I and XVI.)



PROBLEM VIII.

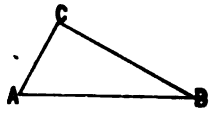
8. *The three sides of triangle given, to construct the triangle.*



Take A B equal to one of the given sides. With A as a centre, with a radius equal to another of the given sides, describe an arc; and with B as a centre, with a radius equal to the third side, describe an arc intersecting the first arc at C. Draw A C and B C, and A C B is evidently the triangle required.

PROBLEM IX.

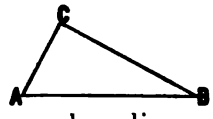
9. *Two sides and the included angle of a triangle given, to construct the triangle.*



Draw A B equal to one of the given sides, at B make the angle A B C equal to the given angle (by Problem V), and take B C equal to the other given side; join A C, and A B C is evidently the triangle required.

PROBLEM X.

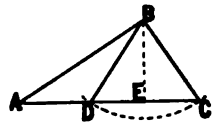
10. *Two angles and a side of a triangle given, to construct the triangle.*



If the angles are not both adjacent to the given sides, find the third angle (by Problem VII.) Then draw A B equal to the given side, and at B make an angle A B C equal to one of the angles adjacent to A B, and at A make an angle B A C equal to the other angle adjacent to A B, and A B C is evidently the triangle required.

PROBLEM XI.

11. *Two sides of a triangle and the angle opposite one of them given, to construct the triangle.*



Draw an indefinite line A C, at A make an angle C A B equal to the given angle, and take A B equal to side adjacent to the given angle; with B as a centre, with a radius equal to the other given side, describe an arc cutting A C. If the angle A is acute.

1st. The given side B C, opposite the given angle, may be less than the other given side, then the arc described from B as a centre will cut A C in two points, C and D, on the same side of A, and, drawing B C and B D, the triangles A B C and A B D (whose angle B D A is the supplement of the angle B C A), both satisfy the given conditions.

2nd. The given side opposite the given angle may be equal to the perpendicular B E; then the arc described from B as a centre will touch A C, and the right triangle A B E is the only one that can satisfy the given conditions.

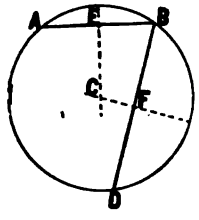
3rd. The side opposite the given angle may be greater than the other given side; then the arc described from B as a centre will cut A C in C, and in another point on the other side of A. In this case there can be but one triangle A B C satisfying given conditions, the triangle formed on the opposite side of A B containing not the given angle, but its supplement.

4th. If the given angle is not acute, the given side opposite the given angle must be greater than the other given side, and, as in the last case above, there can be but one solution.

PROBLEM XII.

12. *To find the centre of a given circumference or of a given arc.*

Let A B D be the given circumference, or arc.



Draw any two chords not parallel to each other, as A B, B D, and bisect these chords by the perpendiculars C E and C F.

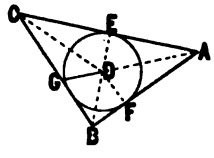
These perpendiculars will intersect at the center of the circumference, or arc. (Theorem XLII).

Scholium. By the same construction a circumference may be made to pass through any three given points; or a circle circumscribed about a given triangle.

PROBLEM XIII.

13. *To inscribe a circle in a given triangle.*

Let A B C be the given triangle.



Bisect any two of its angles. With the point D, where the two bisecting lines meet, as a centre, with a radius equal to the distance of D from any one of the sides, describe a circle, and it will be the one required.

Draw the perpendiculars D E, D F, and D G. The angles at A are equal by construction, and the angles A E D and A F D

are each right angles; therefore the triangles A D E and A F D are mutually equiangular (Theorem XVI), and the hypothenuse A D is common; therefore the triangles are equal (Theorem XVII), and D E = D F. In like manner D E = D G.

Therefore, the circle described from D as a centre with the radius D E will pass through the points F and G; and since the angles at E, F, and G, are right angles, the sides of the triangle A B C are tangents; therefore the circle E F G is inscribed in the triangle A B C.

Scholium. The lines bisecting the angles of a triangle all meet in the same point.

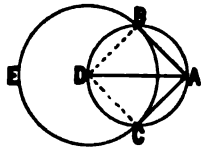
PROBLEM XIV.

14. ***Through a given point to draw a tangent to a given circumference.***

1st. If the given point is in the circumference. Erect a perpendicular to the radius at the given point.

2nd. If the given point is without the circumference.

Join the given point A with the centre C of the given circle B D E; on A C as a diameter describe a circle cutting the given circle in B and D. Draw A B and A D, and each will be tangent to the given circle through the given point.



For drawing the radii C B and C D, the angles B and D are each right angles (Theorem XLV), and therefore A B and A D are tangents to the given circle.

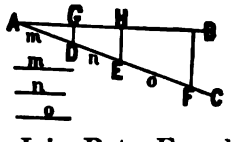
Corollary. The tangents A B and A D are equal.

PROBLEM XV.

15. ***To divide a given line into parts proportional to given lines.***

Let it be required to divide A B into parts proportional to m, n, and o.

Draw at any angle with A B an indefinite line A C. From A cut off A D, D E, and E F respectively equal to m, n, and o. Join B to F and through D and E draw lines parallel to B F. These parallels divide the line as required. (Theorem XLIX).



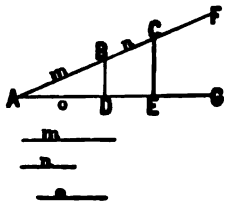
PROBLEM XVI.

16. *To find a fourth proportional to three given lines.*

Let it be required to find a fourth proportional to M, N, and O.

Draw at any angle with the other the indefinite lines A F, and A G.

From A F cut off A B = m, B C = n, and from A G cut off A D = o. Join B D and through C draw C E parallel to B D; then D E is the required fourth proportional (Theorem XLIX).

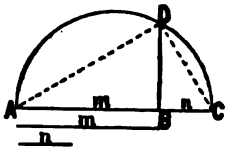


PROBLEM XVII.

17. *To find a mean proportional between two given lines.*

Let it be required to find a mean proportional between m and n.

From an indefinite line cut off A B = M, B C = F; on A C as a diameter describe a semi-circle, and at B draw B D perpendicular to A C. B D is the mean proportional required. Join A D, and D C. (Theorems XLIX and LIX).



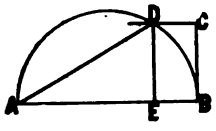
18. Definition. When a line is divided so that one segment is a mean proportional between the whole line and the other segment, it is said to be divided *in extreme and mean ratio.*

PROBLEM XVIII.

19. *To construct a parallelogram, having the sum of its base and altitude given, which shall be equivalent to a given square.*

On A B, the given sum, as a diameter, describe a semi-circumference.

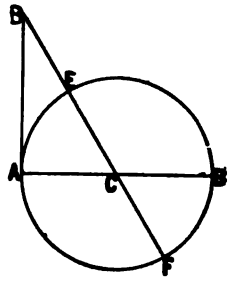
At any point, as B, in A B, draw the perpendicular B C equal to a side of the given square; through C draw C D parallel to A B, cutting the circumference in D; draw D E perpendicular to A B. A E and E B are respectively the base and the altitude of the parallelogram required (Problem XVII).



Scholium. If the side of the square is greater than half the sum of the base and the altitude, the construction is impossible.

PROBLEM XIX.

20. *To construct a parallelogram having the difference between its base and altitude given, which shall be equivalent to a given square.*



On A B the given difference, as a diameter, describe a semi-circumference. At A draw the perpendicular A D equal to a side of the given square; join D with the centre C, and produce D C to F. D F and D E are respectively the base and the altitude of the parallelogram required.

PROBLEM XX.

21. *To construct a square equivalent to a given parallelogram.*

Find a mean proportional between the altitude and base of the given parallelogram (Problem XVII), and it will be a side of the required square.

PROBLEM XXI.

22. *To construct a square equivalent to a given triangle.*

Find a mean proportional between the base and half the altitude (Problem XVII), and it will be a side of the required square.

PROBLEM XXII.

23. *To construct a square equivalent to a given circle.*

Find a mean proportional between the radius and the semi-circumference, and it will be a side of the required square.

PROBLEM XXIII.

24. *To construct a square equivalent to the sum of two given squares.*

Construct a right triangle (Problem IX), with the sides adjacent to the right angle equal respectively to the sides of the given squares; the hypothenuse will be a side of the required square (Theorem LX).

Scholium. By continuing the same process we can find a square equivalent to the sum of any number of given squares.

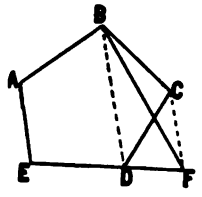
PROBLEM XXIV.

25. *To construct a square equivalent to the difference of two given squares.*

Construct a right triangle (Problem XI), taking as the hypothenuse a side of the greater square, and for one of the sides adjacent to the right angle a side of the other square; the third side of the triangle will be a side of the required square. (Theorem LX.)

PROBLEM XXV.

26. *To construct a triangle equivalent to a given polygon.*

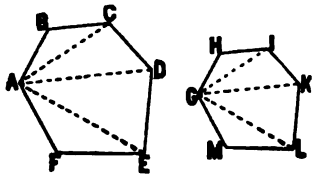


Let A — E be the polygon. Draw B D cutting off the triangle B C D; through C draw C F parallel to B D; join B F, and a polygon A B F E is formed with one side less than the given polygon and equivalent to it. For the triangles B C D and B F D, having the same base and the same altitude, are equivalent; adding to each the common part A B D E, we have A B C D E equivalent to A B F E. In like manner a polygon with one side less can be found equivalent to A B F E, and by continuing the process the sides may be reduced to three, and a triangle obtained equivalent to the given polygon,

Scholium. Since (by Problem XXI) a square can be found equivalent to a given triangle, it is evident that a square can be found equivalent to any polygon.

PROBLEM XXVI.

27. *On a given line to construct a polygon similar to a given polygon.*



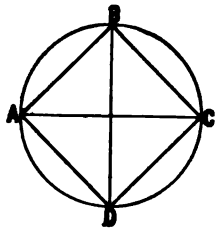
Let A D be the given polygon, and let M L be the given line.

Draw the diagonals A E, A D, and A C. At M and L make the angles G M L and G L M equal

spectively to A F E and A E F, and a triangle G L M will be formed similar to A E F. In like manner on G L construct a triangle similar to A D E; on G K one similar to A C D; on G I one similar to A B C; and the polygons A D, and H G, being composed of the same number of similar triangles similarly situated, are similar. (Theorem LVI.)

PROBLEM XXVII.

28. *To inscribe a square in a given circle.*

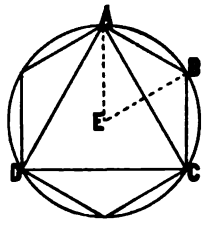


Draw two diameters A C and B D, at right angles to each other; and join A B, B C, C D, and D A. Then A B C D is the required square. (Theorem XLV and XXXIX.)

Corollary. By bisecting the arc A B, B C, C D, and D A, and drawing the chords of these smaller arcs, a regular octagon will be inscribed in the circle. By continuing this bisection regular polygons can be inscribed having the number of their sides 16, 32, 64, and so on.

PROBLEM XXVIII.

29. *To inscribe a regular hexagon in a given circle.*

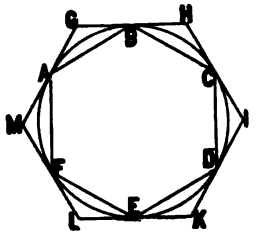


Take A B equal to the radius of the given circle, and it will be a side of the hexagon required for the angle, A B E is equilateral and therefore each of its angles has sixty degrees (Theorem XVI) and is one-sixth of four right angles. Hence A B = B C, etc, (by Theorems XXXVII and XXXIX.)

Corollary. By drawing A C, C D, and D A an equilateral triangle will be inscribed in the circle. By bisecting the arcs A B, B C, etc., and continuing this bisection, as in Problem XXVII, and drawing the chords of these smaller arcs, regular polygons can be inscribed, having the number of their sides 12, 24, 48, 96, and so on.

PROBLEM XXIX.

30. *To circumscribe about a given circle a polygon similar to a given inscribed regular polygon.*



Let A F be the given inscribed polygon. Through the points A, B, C, D, E, F draw tangents to the circumference. Those tangents intersecting will form the polygon required.

For the triangles A G B, B H C, etc., are isosceles (Problem XIV, Corollary;) and as the arcs A B, B C, etc., are equal, the angles G A B, G B A, H B C, H C B, etc., are equal (Theorem XLVI,) therefore, as the bases A B, B C, etc., are equal, those isosceles triangles are equal. Hence the angles G, H, I, K, L, M are equal, and polygon M L is equiangular; and as G B = B H = H C, etc., G H = H I, etc., therefore the polygon M I is equilateral and regular. It is also similar to A F (easily proved from Theorem LVI); and as its sides are tangents it is circumscribed about the circle.

SECTION 2.

TRIGONOMETRY.

DEFINITIONS.

1. **Geometry** shows how to *construct* a triangle and how to find the unknown parts by actual measurement. **Trigonometry** shows how to *compute* the unknown parts from the numerical values of the known parts.

In any triangle there are six parts: three sides and three angles. These six parts are so related that any three, provided one of them is a side, determine the shape and size of the triangle.

2. *Trigonometry*, then, is that branch of mathematics which treats of the numerical computation of *angles* and *triangles*.

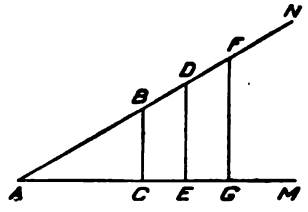


Fig. 1.

3. In deducing rules or formulas to be used in computations, we use what are called **functions** of the angles. This word function is used in mathematics to denote a quantity which depends on some other quantity for its value. Thus the area of a circle depends upon the radius of a circle for its value; hence the area of a circle is a function of the radius.

4. Let M A N be any acute angle. From B, D and F, any points in A N, draw lines perpendicular to A M, forming the right triangles B C A, D E A and F G A. These right triangles are similar, since they are mutually equiangular. The homologous sides are then proportional. Therefore:

$$\frac{CB}{AB} = \frac{ED}{AD} = \frac{GF}{AF} \text{ and } \frac{CB}{AC} = \frac{ED}{AE} = \frac{GF}{AG}, \text{ etc.}$$

These ratios, which depend upon the magnitude of angle A

for their values, are functions of A. There are six such ratios and they are defined and named as follows:

$\frac{CB}{AB}$ = **sine of A** (written ***sin*** A).

$\frac{AC}{AB}$ = **cosine of A** (written *cos* A).

$\frac{CB}{AC}$ = **tangent of A** (written *tan* A).

$\frac{AC}{CB}$ = **cotangent of A** (written *cot* A).

$\frac{AB}{AC}$ = **secant of A** (written *sec* A).

$\frac{AB}{CB}$ = **cosecant of A** (written *csc* A).

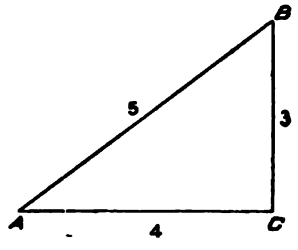


Fig. 2.

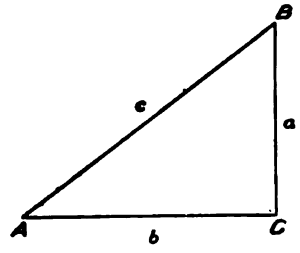


Fig. 3.

Example.— Take the right triangle A C B, Fig. 2, with the sides 3, 4, 5. Then from our definitions:

$\sin A = \frac{3}{5}$ $\quad\quad$ $\cot A = \frac{4}{3}$

$\cos A = \frac{4}{5}$ $\quad\quad$ $\sec A = \frac{5}{4}$

$\tan A = \frac{3}{4}$ $\quad\quad$ $\csc A = \frac{5}{3}$

5. When in Art. 13 to Art. 17 we take up the solution of right triangles, we shall use a triangle lettered as in Fig. 3. A and B are the acute angles, C the right angle, and the opposite sides are represented by the corresponding small letters. That is, the side opposite angle A is a, that opposite B is b, etc.

Then,

$$\left.\begin{aligned}
\sin A &= \frac{a}{c} = \frac{\text{opposite side}}{\text{hypothenuse}} && \text{Rule 1}\\
\cos A &= \frac{b}{c} = \frac{\text{adjacent side}}{\text{hypothenuse}} && \text{" 2}\\
\tan A &= \frac{a}{b} = \frac{\text{opposite side}}{\text{adjacent side}} && \text{" 3}\\
\cot A &= \frac{b}{a} = \frac{\text{adjacent side}}{\text{opposite side}} && \text{" 4}\\
\sec A &= \frac{c}{b} = \frac{\text{hypothenuse}}{\text{adjacent side}} && \text{" 5}\\
\csc A &= \frac{c}{a} = \frac{\text{hypothenuse}}{\text{opposite side}} && \text{" 6}
\end{aligned}\right\} \quad [1]$$

The six ratios are called the *Trigonometric Functions* of angle A.

6. The following definitions must be committed to memory:

The *sine* of an angle is the ratio of the opposite side to the hypothenuse.

The *cosine* of an angle is the ratio of the adjacent side to the hypothenuse.

The *tangent* of an angle is the ratio of the opposite side to the adjacent side.

The *cotangent* of an angle is the ratio of the adjacent side to the opposite side.

The *secant* of an angle is the ratio of the hypothenuse to the adjacent side.

The *cosecant* of an angle is the ratio of the hypothenuse to the opposite side.

7. If we apply these definitions to angle B we have the following:

$$\sin B = \frac{b}{c} \qquad \cot B = \frac{a}{b}$$

$$\cos B = \frac{a}{c} \qquad \sec B = \frac{c}{a}$$

$$\tan B = \frac{b}{a} \qquad \csc B = \frac{c}{b}$$

By inspection of the functions of A and B (which are complementary angles) we see that:

$$\left.\begin{aligned}
\sin A &= \cos B & \cos A &= \sin B\\
\tan A &= \cot B & \cot A &= \tan B\\
\sec A &= \csc B & \csc A &= \sec B
\end{aligned}\right\} \quad [2]$$

EXAMPLES FOR PRACTICE.

Calculate the value of the six functions of angle A in the right triangle A C B, whose sides a, b, c are respectively:

(1) 40, 9, 41.
(2) 8, 15, 17.
(3) 5, 12, 13.
(4) 196, 315, 371.

8. Given one function of an angle to find the other functions. Suppose we have given *sin* A $= \frac{1}{2}$, and the other six functions are required. We construct a right triangle A C B (Fig. 4) calling the opposite side 1 and the hypothenuse 2. We can do this because

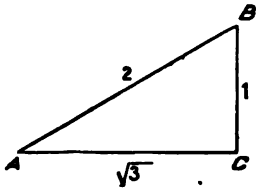


Fig. 4.

$$sin\ A = \frac{opp.\ side}{hyp.} = \frac{1}{2}$$

By Geometry $A\,C = \sqrt{A\,B^2 - C\,B^2} = \sqrt{4-1} = \sqrt{3}$

Hence the adjacent side is $\sqrt{3}$.

Therefore $cos\ A = \frac{\sqrt{3}}{2}$ $sec\ A = \frac{2}{\sqrt{3}}$

$tan\ A = \frac{1}{\sqrt{3}}$

$cot\ A = \sqrt{3}$ $csc\ A = 2$.

EXAMPLES FOR PRACTICE.

Find the values of the five remaining functions, when

1. *tan* A $= \frac{2}{3}$
2. *csc* A $= 4$
3. *sin* A $= \frac{5}{8}$
4. *cot* A $= \frac{4}{7}$

Measurement of Angles. As we learned in "Mensuration," angles are usually expressed in degrees, minutes and seconds. A degree is a ninetieth part of a right angle; a minute is one-sixtieth of a degree, and a second is one-sixtieth of a minute. The symbols used to denote degrees, minutes and seconds are $^\circ\ '\ ''$. Thus forty-six degrees fifteen minutes and six seconds is written $46^\circ\ 15'\ 6''$. In adding two angles, seconds are added to seconds,

minutes to minutes, and degrees to degrees; also, in subtracting two angles, seconds are subtracted from seconds, minutes from minutes, and degrees from degrees.

Example 1.— Add 67° 45′ 16″ and 32° 28′ 52″.

Solution:

$$\begin{array}{r} 67^\circ\ 45'\ 16'' \\ 32^\circ\ 28'\ 52'' \\ \hline 99^\circ\ 73'\ 68'' \end{array}$$

But $68'' = 1'\ 8''$ and $73' = 1^\circ\ 13'$.

Therefore $99^\circ\ 73'\ 68'' = 100^\circ\ 14'\ 8''$ Ans.

Example 2.— Subtract 35° 28′ 14″ from 119° 14′ 25″.

Solution:

$$\begin{array}{r} 119^\circ\ 14'\ 25'' \\ 35^\circ\ 28'\ 14'' \\ \hline \end{array}$$

Since 28′ cannot be taken from 14′ we borrow 1° (= 60′) from 119° and write the angles

$$\begin{array}{r} 118^\circ\ 74'\ 25'' \\ 35^\circ\ 28'\ 14'' \\ \hline 83^\circ\ 46'\ 11'' \end{array}$$ Ans.

Example 3.— In a right triangle one acute angle is 16° 25′ 13″, find the other acute angle.

We know that the sum of the acute angles of a right triangle is equal to one right angle. We write 90° as 89° 59′ 60″, then subtracting the given angle,

$$\begin{array}{r} 89^\circ\ 59'\ 60'' \\ 16^\circ\ 25'\ 13'' \\ \hline 73^\circ\ 34'\ 47'' \end{array}$$ Ans.

EXAMPLES FOR PRACTICE.

1. Add 176° 59′ 24″ and 3° 12′ 56″. Ans. 180° 12′ 20″.
2. Add 56° 28′ 32″ and 39° 43′ 17″. Ans. 96° 11′ 49″.
3. From 84° 19′ 27″ take 44° 20′ 34″. Ans. 39° 58′ 53″.

10. The Functions of 45°. In order to find the functions of 45° take a right isosceles triangle A C B (Fig. 5), each side about the right angle being equal to unity. Then

$$AB = \sqrt{\overline{AC}^2 + \overline{BC}^2} = \sqrt{1+1} = \sqrt{2.} \quad \angle A + \angle B = 90°$$

and as $A = B$ each equals 45°. Therefore, by definition:

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \cot 45° = 1.$$

$$\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \sec 45° = \sqrt{2.}$$

$$\tan 45° = 1 \qquad \csc 45° = \sqrt{2.}$$

In the above, the radical sign is removed from the denominator by multiplying both numerator and denominator by the denominator.

Then, $$\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{2}}{2}.$$

This process is called rationalizing the denominator.

11. Functions of 30° and 60°. To find the functions of 30° and 60°, take an equilateral triangle A B D (Fig. 6). Assume each side equal to 2. Draw A C perpendicular to B D. Then by geometry, B C = C D and angle B A C = angle C A D.

Also $$AC = \sqrt{\overline{AB}^2 - \overline{BC}^2} = \sqrt{4-1} = \sqrt{3.}$$

But if A B D is equilateral, the triangle is equiangular, and each angle is 60°. Hence $\angle ABC = 60°$ and $\angle BAC$ or one-half of $\angle BAD = 30°$.

Applying our definitions to the right triangle A C B:

$$\sin CAB = \sin 30° = \frac{1}{2} \qquad \cot CAB = \cot 30° = \sqrt{3}$$

$$\cos CAB = \cos 30° = \frac{\sqrt{3}}{2} \qquad \sec CAB = \sec 30° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\tan CAB = \tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \csc CAB = \csc 30° = 2.$$

$$\sin CBA = \sin 60° = \frac{\sqrt{3}}{2} \qquad \cot CBA = \cot 60° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cos CBA = \cos 60° = \frac{1}{2} \qquad \sec CBA = \sec 60° = 2$$

$$\tan CBA = \tan 60° = \sqrt{3} \qquad \csc CBA = \csc 60° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

FUNDAMENTAL THEOREMS.

12. Since $\sin A = \frac{a}{c}$ and $\csc A = \frac{c}{a}$: (Fig. 3)

Since $\cos A = \frac{b}{c}$ and $\sec A = \frac{c}{b}$

Since $\tan A = \frac{a}{b}$ and $\cot A = \frac{b}{a}$

It is evident that

$$\left.\begin{aligned} \sin A &= \frac{1}{\csc A}; \quad \cos A = \frac{1}{\sec A}; \quad \tan A = \frac{1}{\cot A} \\ \csc A &= \frac{1}{\sin A}; \quad \sec A = \frac{1}{\cos A}; \quad \cot A = \frac{1}{\tan A} \end{aligned}\right\} \quad [3]$$

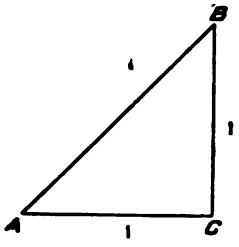


Fig. 5.

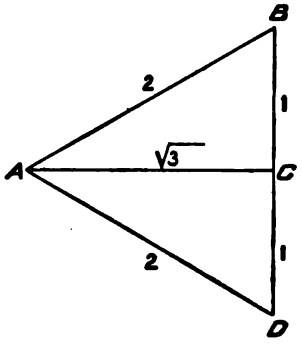


Fig. 6.

13. In the right triangle A B C, Fig. 3,

By Geometry $\quad \overline{BC}^2 + \overline{AC}^2 = \overline{AB}^2$

Or $\quad a^2 + b^2 = c^2$

Dividing by c^2 $\quad \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2}$

That is $\quad \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$

Whence by definition $(\sin A)^2 + (\cos A)^2 = 1$

This is usually written,

$$\sin^2 A + \cos^2 A = 1 \qquad [4]$$

(It is read, sine squared A plus cosine squared A equals one.)

14. By our definition $tan\ A = \frac{a}{b}$

Divide numerator and denominator by c

Then $$tan\ A = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{sin\ A}{cos\ A} \qquad [5]$$

Similarly, $$cot\ A = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{cos\ A}{sin\ A} \qquad [6]$$

15. In triangle A B C, Fig. 3,

By Geometry $$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$$

Or $$c^2 = b^2 + a^2$$

Divide by b^2, $$\frac{c^2}{b^2} = 1 + \frac{a^2}{b^2}$$

That is $$\left(\frac{c}{b}\right)^2 = 1 + \left(\frac{a}{b}\right)^2$$

16. By definition $$sec^2\ A = 1 + tan^2\ A \qquad [7]$$

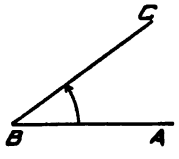
Similarly divide $c^2 = b^2 + a^2$ by a^2

$$\frac{c^2}{a^2} = \frac{b^2}{a^2} + 1$$

Whence by definition $$csc^2\ A = cot^2\ A + 1 \qquad [8].$$

FUNCTIONS OF ANY ANGLE.

17. In Art. 4, definitions for the functions of *acute* angles only were given. Before we can define the functions of *any* angle we must consider angles themselves.

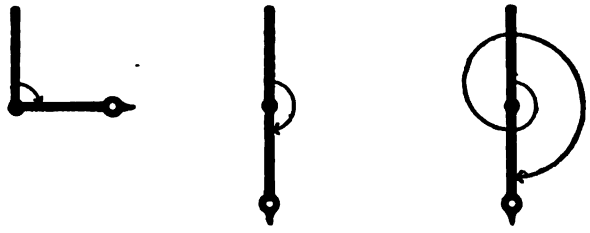


In Trigonometry an angle is considered as generated by revolving a line from some initial position (called initial line) to some final position (called terminal line). Thus the angle A B C may be generated by revolving a line from the position B A to the position B C.

If the minute hand of a clock makes $\frac{1}{4}$ of a revolution, it turns through one right angle; if it makes one-half a revolution,

it turns through two right angles; if one and one-half revolutions, it turns through six right angles. Thus :

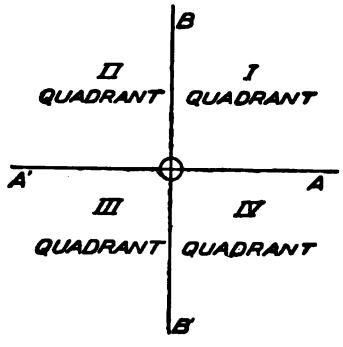
By revolving a sufficient number of times it is evident that an angle of any magnitude may be formed.



18. Positive Angles. If the direction of revolution is *opposite* to that of the hands of a clock, the angle is said to be *positive*.

Negative Angles. If the direction of revolution is the same as that of the hands of a clock, the angle is said to be negative. The angles in Art. 17 are all negative.

Example.— Through how many right angles does the hour hand of a clock move in $8\frac{1}{2}$ hours? The minute hand?

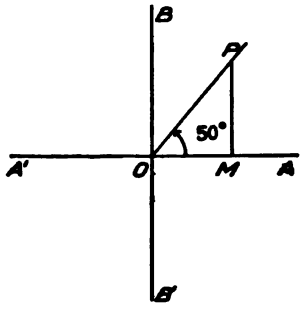


Example.— If the fly wheel of an engine makes 60 revolutions per minute, through how many right angles does a spoke of the fly wheel revolve in one second?

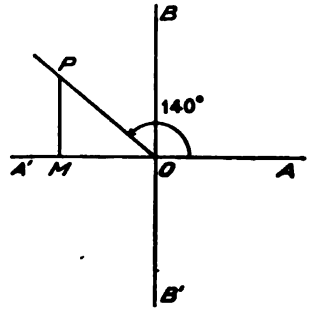
19. Let B B′ be drawn perpendicular to A A′, intersecting it at O. The plane of the page is then divided into four parts

called quadrants. A O B is called the first quadrant, B O A′ is called the second quadrant, A′ O B′ is called the third quadrant, and B′ O A is called the fourth quadrant.

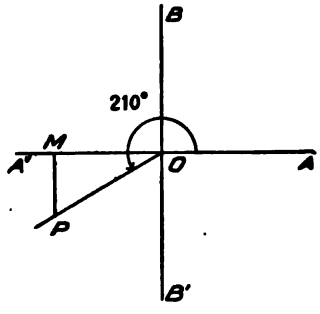
FIRST QUADRANT



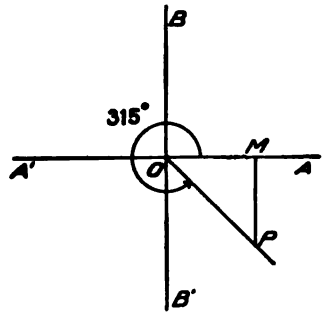
SECOND QUADRANT



THIRD QUADRANT



FOURTH QUADRANT



An angle is said to lie in the quadrant in which its terminal line lies.

20. Construct an angle A O P in each of the four quadrants. Draw P M perpendicular to A A′, meeting it at M, forming the triangle M O P. The sides of this triangle determine the functions of the angle. If the perpendicular M P is drawn above A A′, it is considered *positive*; if below A A′, it is considered *negative*. If the base O M is drawn to the right of B B′, it is considered positive; if drawn to the left of B B′, it is considered negative. O P is always positive. Hence, some of the functions of angles in the second, third and fourth quadrants are negative.

21. The trigonometric functions of any angle then are,

$$\sin A\,O\,P = \frac{\text{perpendicular}}{\text{hypothenuse}} \quad \begin{cases} + \text{ in I and II.} \\ - \text{ in III and IV.} \end{cases}$$

$$\cos A\,O\,P = \frac{\text{base}}{\text{hypothenuse}} \quad \begin{cases} + \text{ in I and IV.} \\ - \text{ in II and III.} \end{cases}$$

$$\tan A\,O\,P = \frac{\text{perpendicular}}{\text{base}} \quad \begin{cases} + \text{ in I and III.} \\ - \text{ in II and IV.} \end{cases}$$

$$\cot A\,O\,P = \frac{\text{base}}{\text{perpendicular}} \quad \begin{cases} + \text{ in I and III.} \\ - \text{ in II and IV.} \end{cases}$$

$$\sec A\,O\,P = \frac{\text{hypothenuse}}{\text{base}} \quad \begin{cases} + \text{ in I and IV.} \\ - \text{ in II and III.} \end{cases}$$

$$\operatorname{csc} A\,O\,P = \frac{\text{hypothenuse}}{\text{perpendicular}} \quad \begin{cases} + \text{ in I and II.} \\ - \text{ in III and IV.} \end{cases}$$

The signs of the functions may be tabulated thus:

Quadrant	I	II	III	IV
Sine and cosecant	+	+	—	—
Tangent and cotangent	+	—	+	—
Secant and cosine	+	—	—	+

EXAMPLES FOR PRACTICE.

1. Construct the following angles: 150°, 225°, 270°, 300°, 400°, —120°, —45°, 840°.
2. The terminal side of 650° lies in what quadrant?
3. The terminal side of 430° lies in what quadrant?

FUNCTIONS OF 0°, 90°, 180°, 270°.

22. Functions of 0°. Let Fig. 7 represent a circle with a radius equal to unity. Let A O P be an angle in the first quadrant.

Then $\sin A\,O\,P = \frac{M\,P}{O\,P} = \frac{M\,P}{1} = M\,P$

$\cos A\,O\,P = \frac{O\,M}{O\,P} = \frac{O\,M}{1} = O\,M$

Let O P be revolved towards O A, the angle A O P growing smaller and smaller and approaching zero degrees as O P approaches the position O A. Then M P approaches zero and O M approaches O A.

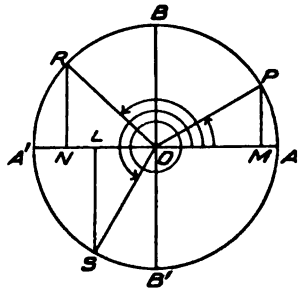


Fig. 7.

$$\sin 0^\circ = \frac{MP}{OP} = \frac{0}{1} = 0; \qquad \cos 0^\circ = \frac{OM}{OP} = \frac{1}{1} = 1$$

Therefore $\sin 0^\circ = 0$

and $\cos 0^\circ = OA = 1.$

By formula [5] $\tan 0^\circ = \dfrac{\sin 0^\circ}{\cos 0^\circ} = \dfrac{0}{1} = 0$

By formula [6] $\cot 0^\circ = \dfrac{\cos 0^\circ}{\sin 0^\circ} = \dfrac{1}{0} = \infty$

By formula [3] $\sec 0^\circ = \dfrac{1}{\cos 0^\circ} = \dfrac{1}{1} = 1$

By formula [3] $\csc 0^\circ = \dfrac{1}{\sin 0^\circ} = \dfrac{1}{0} = \infty.$

23. Functions of 90°. Let O P (Fig. 7) be revolved until it coincides with O B. Then $\angle$ A O P becomes 90°, O M = O and M P = O B.

Therefore $\sin 90^\circ = OB = 1$

$\cos 90^\circ = 0$

$\tan 90^\circ = \dfrac{\sin 90^\circ}{\cos 90^\circ} = \dfrac{1}{0} = \infty$

$\cot 90^\circ = \dfrac{\cos 90^\circ}{\sin 90^\circ} = \dfrac{0}{1} = 0$

$\sec 90^\circ = \dfrac{1}{\cos 90^\circ} = \dfrac{1}{0} = \infty$

$\csc 90^\circ = \dfrac{1}{\sin 90^\circ} = \dfrac{1}{1} = 1.$

24. Functions of 180°. Let A O R (Fig. 7) be an angle in the second quadrant.

$$\text{Then } \sin AOR = \frac{NR}{OR} = \frac{NR}{1} = NR$$

$$\cos AOR = \frac{ON}{OR} = \frac{ON}{1} = ON$$

Let O R revolve until it coincides with O A′, then $\angle$ A O R becomes $\angle$ A O A′ or 180°, N R = O and O N = O A′ = − 1.

NOTE. ∞ is the symbol used to denote an infinite quantity, that is, a quantity larger than any assignable quantity. Let us consider the series: $\frac{a}{c}, \frac{10a}{c}, \frac{100a}{c}, \frac{1{,}000a}{c}$. It is evident that if the series is continued, the absolute value of the numerator may be made greater than any assigned positive quantity, and as the denominator remains the same, the absolute value of the fraction may be made greater than any assigned positive quantity. Similarly it may be proved that the value of a fraction becomes very large when its denominator becomes very small.

(See Art. 20.)

Therefore $\sin 180° = 0$

$\cos 180° = O A = -1$

$\tan 180° = \frac{\sin 180°}{\cos 180°} = \frac{0}{-1} = 0$

$\cot 180° = \frac{\cos 180°}{\sin 180°} = \frac{-1}{0} = \infty$

$\sec 180° = \frac{1}{\cos 180°} = \frac{1}{-1} = -1$

$\csc 180° = \frac{1}{\sin 180°} = \frac{1}{0} = \infty$

25. Functions of 270°. Let A O S be an angle in the third quadrant.

$$\text{Then } \sin A O S = \frac{L S}{O S} = \frac{L S}{1} = L S$$

$$\cos A O S = \frac{O L}{O S} = \frac{O L}{1} = O L$$

Let O S revolve until it coincides with O B′, then $\angle A O S$ becomes $\angle A O B'$, or 270°, O L = O, and L S = O B′.

Therefore $\sin 270° = O B' = -1$

$\cos 270° = 0$

$\tan 270° = \frac{\sin 270°}{\cos 270°} = \frac{-1}{0} = \infty$

$\cot 270° = \frac{\cos 270°}{\sin 270°} = \frac{0}{-1} = 0$

$\sec 270° = \frac{1}{\cos 270°} = \frac{1}{0} = \infty$

$\csc 270° = \frac{1}{\sin 270°} = \frac{1}{-1} = -1$

The foregoing results may be expressed in a tabular form as follows:

Angle.	Sin.	Cos.	Tan.	Cot.	Sec.	Csc.
0°	0	1	0	∞	1	∞
90°	1	0	∞	0	∞	1
180°	0	−1	0	∞	−1	∞
270°	−1	0	∞	0	∞	−1

RELATIONS BETWEEN FUNCTIONS OF ANGLES.*

26. Functions of $(-x)$, $(180°-x)$, $(180°+x)$, $(360°-x)$.

Let $\angle$ A O P, in Figs. 8, 9, 10 and 11 represent angle x and $\angle$ A O P′ represent $-x$.

If P M P′ is drawn perpendicular to A A′ the right triangles M O P and M O P′ have a side and acute angle of one equal respectively to a side and acute angle of the other and are therefore equal.

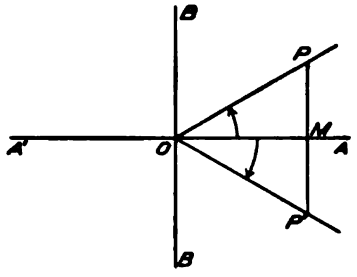


Fig. 8.

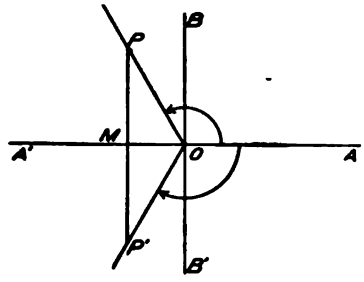


Fig. 9.

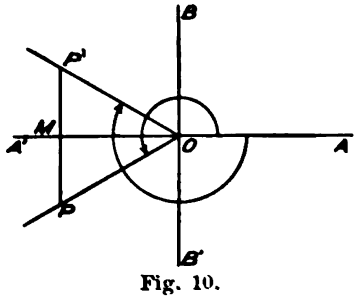


Fig. 10.

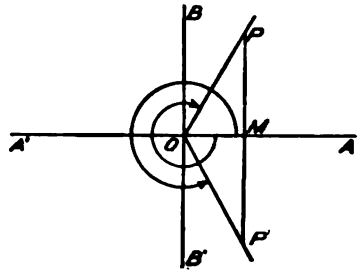


Fig. 11.

Applying the definitions given in Art. 21, noting carefully the signs, we have,

$$\left.\begin{array}{lll} \sin(-x) = -\sin x & \tan(-x) = -\tan x & \sec(-x) = -\sec x \\ \cos(-x) = -\cos x & \cot(-x) = -\cot x & \csc(-x) = -\csc x \end{array}\right\} \quad [9]$$

27. If x is an acute angle, $180° - x$ lies in the second quadrant, $180° + x$ in the third quadrant and $360° - x$ in the fourth quadrant.

In Figs. 12, 13 and 14 construct A O P′ equal to angle x; make O P′ = O P and draw M P and M′ P′ perpendicular to A A′.

* The following 30 Articles are optional.

Then the right triangles P M O and P′ M′ O are equal, since the hypothenuse and acute angle of one equal the hypothenuse and acute angle of the other.

Applying the definitions given in Art. 21, noting carefully the signs, we have,

A O P = 180° − x

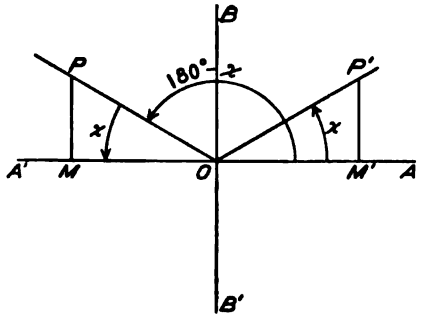


Fig. 12.

A O P = 180° + x

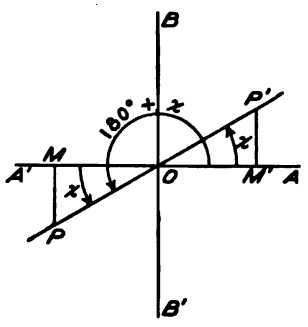


Fig. 13.

A O P = 360° − x

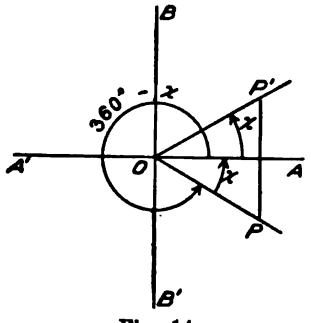


Fig. 14.

$$\left\{\begin{array}{ll} \sin(180^\circ - x) = \sin x & \cot(180^\circ - x) = -\cot x \\ \cos(180^\circ - x) = -\cos x & \sec(180^\circ - x) = -\sec x \\ \tan(180^\circ - x) = -\tan x & \csc(180^\circ - x) = -\csc x \end{array}\right\} [10]$$

Also

$$\left\{\begin{array}{ll} \sin(180^\circ + x) = -\sin x & \cot(180^\circ + x) = \cot x \\ \cos(180^\circ + x) = -\cos x & \sec(180^\circ + x) = -\sec x \\ \tan(180^\circ + x) = \tan x & \csc(180^\circ + x) = -\csc x \end{array}\right\} [11]$$

Also

$$\left.\begin{array}{ll} \sin(360°-x) = -\sin x & \cot(360°-x) = -\cot x \\ \cos(360°-x) = -\cos x & \sec(360°-x) = \sec x \\ \tan(360°-x) = -\tan x & \csc(360°-x) = -\csc x \end{array}\right\} [12]$$

28. To find a function of an angle greater than 360°, subtract a multiple of 360° from the angle, the function remaining the same.

$$\sin 890° = \sin(890° - 720°) = \sin 170°$$
$$= \sin(180° - 10°) = \sin 10°$$
$$\tan 480° = \tan(480° - 360°) = \tan 120°$$
$$= \tan(180° - 60°) = -\tan 60° = -\sqrt{3}$$
$$\csc(-120°) = -\csc 120° = -\csc(180° - 60°) = -\frac{2}{\sqrt{3}}$$

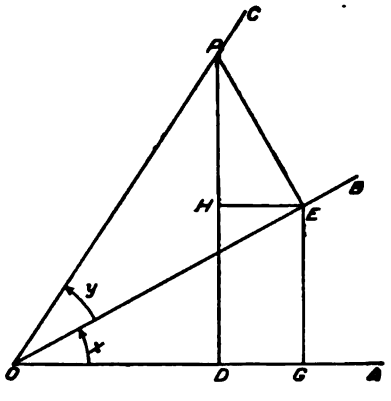
EXAMPLES FOR PRACTICE.

1. $\cos 495°$
2. $\tan 330°$
3. $\sec 225°$
4. $\sin 150°$
5. $\cos(-60°)$

29. Several formulas for the functions of one angle have been derived. We shall now prove four fundamental formulas for functions of two angles and from them derive others.

These four formulas are:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \quad [13]$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y \quad [14]$$
$$\sin(x-y) = \sin x \cos y - \cos x \sin y \quad [15]$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y \quad [16]$$



30. Let $\angle AOB = x$ and $\angle BOC = y$, then $\angle AOC = (x+y)$. From P, any point in O C, draw P D and P E perpendicular to O A and O B respectively. From E draw E G and E H perpendicular to O A and P D respectively. Since P E is perpendicular to O B, and P H is perpendicular to O A, angle H P E is equal to angle A O B or x.

Now $\sin(x+y) = \frac{DP}{OP} = \frac{DH+HP}{OP} = \frac{GE}{OP} + \frac{HP}{OP}$

But $\frac{GE}{OP} = \frac{GE}{OE} \times \frac{OE}{OP} = \sin x \cos y$

And $\frac{HP}{OP} = \frac{HP}{EP} \times \frac{EP}{OP} = \cos x \sin y$

Therefore $\sin(x+y) = \sin x \cos y + \cos x \sin y$ [13]

Similarly $\cos(x+y) = \frac{OD}{OP} = \frac{OG-DG}{OP} = \frac{OG}{OP} - \frac{HE}{OP}$

But $\frac{OG}{OP} = \frac{OG}{OE} \times \frac{OE}{OP} = \cos x \cos y$

And $\frac{HE}{OP} = \frac{HE}{EP} \times \frac{EP}{OP} = \sin x \sin y$

Therefore $\cos(x+y) = \cos x \cos y - \sin x \sin y$ [14]

31. We have merely proved that these formulas hold true when x and y, and $x+y$ are acute. It can be easily proved by means of diagrams and equations that they are also true for all values of x and y, but the proof will not be taken up here.

32. Substituting $-y$ for y in formula [13], it becomes

$$\sin[x+(-y)] = \sin x \cos(-y) + \cos x \sin(-y)$$

But $\cos(-y) = \cos y$ and $\sin(-y) = -\sin y$ by [9]

Therefore $\sin(x-y) = \sin x \cos y - \cos x \sin y$ [15]

Also substituting $-y$ for y in formula [14], it becomes

$$\cos[x+(-y)] = \cos x \cos(-y) - \sin x \sin(-y)$$
$$= \cos x \cos y + \sin x \sin y \quad [16]$$

33. $\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$

Dividing both numerator and denominator by $\cos x \cos y$, this becomes

$$\tan(x+y) = \frac{\frac{\sin x \cos y}{\cos x \cos y} + \frac{\cos x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y}{\cos x \cos y} - \frac{\sin x \sin y}{\cos x \cos y}}$$

34. Cancelling the equal quantities and introducing the formulas $\tan x = \frac{\sin x}{\cos x}$ and $\tan y = \frac{\sin y}{\cos y}$, we finally have

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \qquad [17]$$

Similarly

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} \qquad [18]$$

35. In formula [13] substituting x for y,

$$\sin(x+x) = \sin x \cos x + \cos x \sin x$$

or
$$\sin 2x = 2 \sin x \cos x \qquad [19]$$

36. In formula [14] substituting x for y,

$$\cos(x+x) = \cos x \cos x - \sin x \sin x$$

or
$$\cos 2x = \cos^2 x - \sin^2 x \qquad [20]$$

37. If now we substitute for $\sin^2 x$ its equivalent, $1 - \cos^2 x$ (formula [4]), we have,

$$\cos 2x = \cos^2 x - (1 - \cos^2 x)$$
$$= 2\cos^2 x - 1 \qquad [22]$$

38. If in [20] we substitute for $\cos^2 x$ its equivalent, $1 - \sin^2 x$, we have

$$\cos 2x = 1 - \sin^2 x - \sin^2 x$$
$$= 1 - 2\sin^2 x \qquad [23]$$

39. In formula [17] substitute x for y,

$$\tan(x+x) = \frac{\tan x + \tan x}{1 - \tan x \tan x}$$

or
$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x} \qquad [24]$$

40. From formula [22], by transposing terms,

$$2\cos^2 x = 1 + \cos 2x$$

or
$$2\cos^2 \tfrac{1}{2}x = 1 + \cos x \qquad [25]$$

41. From formula [23], by transposing terms,

$$2\sin^2 x = 1 - \cos 2x$$

or
$$2\sin^2 \tfrac{1}{2}x = 1 - \cos x \qquad [26]$$

42. Dividing [26] by [25],

$$\frac{2\sin^2 \frac{1}{2}x}{2\cos^2 \frac{1}{2}x} = \frac{1 - \cos x}{1 + \cos x}$$

$$\tan^2 \tfrac{1}{2}x = \frac{1 - \cos x}{1 + \cos x}$$

$$\tan \tfrac{1}{2} x = \sqrt{\frac{1 - \cos x}{1 + \cos x}} \qquad [27]$$

43. From formulas [13] to [16] we obtain

$$\sin (x + y) + \sin (x - y) = 2 \sin x \cos y$$
$$\sin (x + y) - \sin (x - y) = 2 \cos x \sin y$$
$$\cos (x + y) + \cos (x - y) = 2 \cos x \cos y$$
$$\cos (x + y) - \cos (x - y) = -2 \sin x \sin y$$

Let $a = (x + y)$

and $b = (x - y)$

Then by adding and subtracting we get

$$x = \tfrac{1}{2} (a + b) \text{ and } y = \tfrac{1}{2} (a - b)$$

Substituting,

$$\sin a + \sin b = 2 \sin \tfrac{1}{2} (a + b) \cos \tfrac{1}{2} (a - b) \qquad [28]$$
$$\sin a - \sin b = 2 \cos \tfrac{1}{2} (a + b) \sin \tfrac{1}{2} (a - b) \qquad [29]$$
$$\cos a + \cos b = 2 \cos \tfrac{1}{2} (a + b) \cos \tfrac{1}{2} (a - b) \qquad [30]$$
$$\cos a - \cos b = -2 \sin \tfrac{1}{2} (a + b) \sin \tfrac{1}{2} (a - b) \qquad [31]$$

These formulas greatly simplify the solution of oblique triangles in case two sides and the included angle are given, or if the three sides are known.

44. It can be proved that in any triangle

$$\frac{a}{b} = \frac{\sin A}{\sin B} \qquad \text{(See Art 87.)}$$

45. By Geometry (Composition and Division),

$$\frac{a + b}{a - b} = \frac{\sin A + \sin B}{\sin A - \sin B}$$

46. By formulas [28] and [29],

$$\frac{a + b}{a - b} = \frac{2 \sin \tfrac{1}{2} (A + B) \cos \tfrac{1}{2} (A - B)}{2 \cos \tfrac{1}{2} (A + B) \sin \tfrac{1}{2} (A - B)}$$

$$= \frac{\sin \tfrac{1}{2} (A + B)}{\cos \tfrac{1}{2} (A + B)} \times \frac{1}{\dfrac{\sin \tfrac{1}{2} (A - B)}{\cos \tfrac{1}{2} (A - B)}}$$

$$= \frac{\tan \tfrac{1}{2} (A + B)}{\tan \tfrac{1}{2} (A - B)} \qquad [32]$$

This formula is used when a, b and C of a triangle are given.

47. By advancing the letters,

$$\frac{b+c}{b-c} = \frac{\tan \frac{1}{2}(B+C)}{\tan \frac{1}{2}(B-C)}$$

$$\frac{c+a}{c-a} = \frac{\tan \frac{1}{2}(C+A)}{\tan \frac{1}{2}(C-A)}$$

48.

In Fig. 15

$AD = b \cos A$
$DB = AB - AD$
$= c - b \cos A$
$CD = b \sin A$

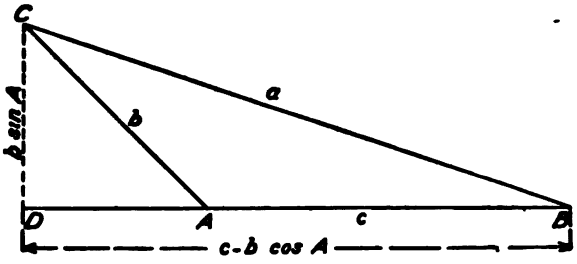


Fig. 15.

In Fig. 16

$AD = b \cos (180° - A)$
$= b \cos A$ by [10]
$DB = AB + AD$
$= c + b \cos A$
$CD = b \sin 180° + A$
$= b \sin A$ by [11]

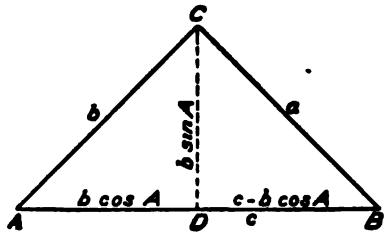


Fig. 16.

49. Now in both Fig. 15 and Fig 16

$$\overline{BC}^2 = \overline{BD}^2 + \overline{DC}^2$$
$$a^2 = (c - b \cos A)^2 + (b \sin A)^2$$
$$= c^2 - 2bc \cos A + b^2 \cos^2 A + b^2 \sin^2 A$$
$$= c^2 - 2bc \cos A + b^2 (\cos^2 A + \sin^2 A)$$
$$= c^2 + b^2 - 2bc \cos A$$

$$\text{or } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

50. The above equation would enable us to find angle A if the three sides are given, but it is not in a form suitable for use with logarithms.

51. From formula [26]

$$2\sin^2\tfrac{1}{2}A = 1 - \cos A = 1 - \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{2bc - b^2 - c^2 + a^2}{2bc}$$

$$= \frac{a^2 - (b-c)^2}{2bc}$$

$$= \frac{(a-b+c)(a+b-c)}{2bc}$$

Let $2s = a+b+c$, then $(a-b+c) = 2(s-b)$, and $(a+b-c) = 2(s-c)$

Substituting, $2\sin^2\tfrac{1}{2}A = \dfrac{2(s-b)(s-c)}{bc}$

Hence $$\sin\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}} \qquad [33]$$

52. From formula [25],

$$2\cos^2\tfrac{1}{2}A = 1 + \cos A = 1 + \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{2bc + b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(b+c)^2 - a^2}{2bc}$$

$$= \frac{(b+c+a)(b+c-a)}{2bc}$$

$$= \frac{2s(s-a)}{bc}$$

Hence $$\cos\tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}} \qquad [34]$$

53. Dividing [33] by [34],

$$\frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A} = \tan\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \qquad [35]$$

54. Similarly

$$\tan\tfrac{1}{2}B = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$$

$$\tan\tfrac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

These formulas are used to find the angles when the three sides are given.

55. In Fig. 15 and Fig. 16 of Art. 48 we see that if K is the area of the triangle,

$$2\,K = A\,B \times C\,D$$
$$= b\,c\,\sin A \qquad [36]$$

If the three sides are given

$$2\,K = 2\,b\,c\,\sin\tfrac{1}{2}A\,\cos\tfrac{1}{2}A \qquad \text{by } [19]$$

Substituting the values of $\sin\frac{1}{2}A$ and $\cos\frac{1}{2}A$ given in formulas [33] and [34],

$$K = b\,c\,\sqrt{\frac{(s-b)(s-c)}{b\,c}}\,\sqrt{\frac{s(s-a)}{b\,c}}$$
$$= \sqrt{s(s-a)(s-b)(s-c)} \qquad [37]$$

This formula is used to find the area of a triangle when the three sides are given.

LOGARITHMS.

56. The logarithm of a number, N, is the exponent of the power to which a fixed number, called the base, must be raised in order to produce N.

Thus, if $3^4 = 81$, $\log_3 81 = 4$; or, in words, the logarithm of 81 to base 3 is 4.

What is $\log_2 8$? We know $2^3 = 8$ $\therefore$ $\log_2 8 = 3$

What is $\log_9 3$? We know $9^{\frac{1}{2}} = 3$ $\therefore$ $\log_9 3 = \frac{1}{2}$

EXAMPLES FOR PRACTICE.

1. Find the $\log_3 27$. Ans. 3.
2. Find the $\log_{10} 100$. Ans. 2.
3. Find the $\log_4 64$. Ans. 3.

57. All logarithms of numbers which have the same base are said to belong to the same *system*. The system of which 10 is the base is called the common system. This is the system most frequently used in computation.

58. In any system, the logarithm of the base itself is 1.

For $10^1 = 10$ $\therefore \log_{10} 10 = 1$
$3^1 = 3$ $\therefore \log_3 3 = 1$
$a^1 = a$ $\therefore \log_a a = 1$

59. In any system the logarithm of 1 is 0.

For $10^0 = 1$ $\therefore \log_{10} 1 = 0$
$3^0 = 1$ $\therefore \log_3 1 = 0$
$a^0 = 1$ $\therefore \log_a 1 = 0$

60. Logarithms are used largely to facilitate computation in problems involving multiplication, division, involution and evolution of numbers. Their great usefulness depends on the four following principles.

61. In any system, the logarithm of the product of two or more numbers is equal to the sum of the logarithms of the separate numbers.

Assume $a^x = m$ (1)
And $a^y = n$ (2)
Then by § 56 $\log_a m = x$, $\log_a n = y$

Multiplying equations (1) and (2),

$$(a^x)\ (a^y) = m\,n$$

Or $$a^{x+y} = m\,n$$

$$\therefore \log_a m\,n = x + y$$

Substituting for x and y their values,

$$\log_a m\,n = \log_a m + \log_a n$$

62. In any system the logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator.

Assume $a^x = m$ (1) And $a^y = n$ (2) } Then by § 56 { $\log_a m = x$, $\log_a n = y$

Divide equation (1) by equation (2),

$$\frac{a^x}{a^y} = \frac{m}{n}$$

Or $$a^{x-y} = \frac{m}{n}$$

Then $$\log_a \frac{m}{n} = x - y$$

Substituting for x and y their values,

$$\log_a \frac{m}{n} = \log_a m - \log_a n$$

63. In any system the logarithm of the power of a number is equal to the exponent of the power multiplied by the logarithm of the number.

Assume $a^x = m$ (1), then $\log_a m = x$.

Raising both members of (1) to the p^{th} power

$$a^{px} = m^p$$

Therefore by definition

$$\log_a (m^p) = p\,x$$

Substituting for x its value,

$$\log_a (m^p) = p \log_a m$$

64. The logarithm of a root of a number is equal to the logarithm of the number divided by the index of the root.

By algebra, $$\sqrt[s]{m} = m^{\frac{1}{s}}$$

Therefore, by Art. 63,

$$\log_a \sqrt[s]{m} = \log_a m^{\frac{1}{s}} = \frac{1}{s}\log_a m = \frac{\log_a m}{s}$$

65. A few examples will illustrate these principles.

Given that log 2 = .301030, log 3 = .477121, log 5 = .698970 and log 7 = .845098.

(1) Find log 45.

$$\begin{aligned}\log 45 &= \log (3 \times 3 \times 5)\\ &= \log 3 + \log 3 + \log 5\\ &= 2 \times \log 3 + \log 5\\ &= .954242 + .698970\\ &= 1.653212.\end{aligned}$$

(2) Find log $\frac{14}{5}$.

$$\begin{aligned}\log \tfrac{14}{5} &= \log 14 - \log 5\\ &= (\log 2 + \log 7) - \log 5\\ &= .301030 + .845098 - .698970\\ &= 1.146128 - .698970 = .447158\end{aligned}$$

(3) Find log $(7^{\frac{3}{4}} \times 2^{\frac{1}{2}})$.

$$\begin{aligned}\log (7^{\frac{3}{4}} \times 2^{\frac{1}{2}}) &= \log 7^{\frac{3}{4}} + \log 2^{\frac{1}{2}}\\ &= \tfrac{3}{4}\log 7 + \tfrac{1}{2}\log 2\\ &= \frac{3 \times (.845098)}{4} + \frac{.301030}{2}\\ &= .633824^* + .150515\\ &= .784339\end{aligned}$$

(4) Find log $\frac{\sqrt[4]{5}}{\sqrt[3]{2}}$.

$$\begin{aligned}\log \frac{\sqrt[4]{5}}{\sqrt[3]{2}} &= \log \sqrt[4]{5} - \log \sqrt[3]{2}\\ &= \tfrac{1}{4}\log 5 - \tfrac{1}{3}\log 2\\ &= \frac{.698970}{4} - \frac{.301030}{3}\\ &= .174743 - .100343\\ &= .074400\end{aligned}$$

*Note. Whenever in dividing a logarithm by a number a remainder of ½ or more is obtained, the last figure of the quotient is increased by 1.

EXAMPLES FOR PRACTICE.

Find the logarithms of the following:

(5) $\log 35$ Ans. 1.544068.

(6) $\log 42$ Ans. 1.623249.

(7) $\log \frac{25}{7}$ Ans. .552842.

(8) $\log \sqrt{15}$ Ans. .588046.

(9) $\log (2^{\frac{2}{3}} \times 5^{\frac{1}{2}})$ Ans. .550172.

(10) $\log (7\sqrt[3]{6})$ Ans. 1.104482.

(11) $\log \frac{\sqrt{50}}{3}$ Ans. .372864.

66. We will now write the common logarithms of certain integral powers of 10 and indicate between what limits the logarithms of certain other numbers must lie.

$1000 = 10^3$ $\therefore \log 1000 = 3$

$796 = 10^{2 + \text{a decimal}}$ $\therefore \log 796 = 2 + \text{a decimal}$

$100 = 10^2$ $\therefore \log 100 = 2$

$53 = 10^{1 + \text{a decimal}}$ $\therefore \log 53 = 1 + \text{a decimal}$

$10 = 10^1$ $\therefore \log 10 = 1$

$8 = 10^{0 + \text{a decimal}}$ $\therefore \log 8 = 0 + \text{a decimal}$

$1 = 10^0$ $\therefore \log 1 = 0$

$0.77 = 10^{-1 + \text{a decimal}}$ $\therefore \log 0.77 = 9 + \text{a decimal} - 10$

$0.1 = \frac{1}{10} = 10^{-1}$ $\therefore \log 0.1 = -1 \text{ or } 9 - 10$

$0.079 = 10^{-2 + \text{a decimal}}$ $\therefore \log 0.079 = 8 + \text{a decimal} - 10$

$0.01 = \frac{1}{100} = 10^{-2}$ $\therefore \log 0.01 = -2 \text{ or } 8 - 10$

$0.00274 = 10^{-3 + \text{a decimal}}$ $\therefore \log 0.00274 = 7 + \text{a decimal} - 10$

$0.001 = \frac{1}{1000} = 10^{-3}$ $\therefore \log 0.001 = -3 \text{ or } 7 - 10$

67. From Art. 66 we see that the logarithm of a number which is not an integral power of 10 is an integer plus a decimal. Thus, 796 is between 10^2 and 10^3, therefore its logarithm is between 2 and 3, and is $2 +$ a decimal. Also, 0.079 is between 10^{-2} and 10^{-1}, therefore its logarithm is between -2 and -1 and is $-2 +$ a decimal. But -2 is represented by $8 - 10$,

so that this logarithm would be written 8 + a decimal — 10. Certain advantages are gained by using 8 — 10 instead of — 2, as will be explained later on.

68. The integral part of the logarithm is called its *characteristic*, and the decimal part its *mantissa*. The characteristic, when the number is not an integral power of 10, may be found by observing between what adjacent integral powers of 10 the number lies.

69. From Art. 66 we see that the logarithm of a number between

1 and 10 is equal to 0 + a decimal
10 and 100 is equal to 1 + a decimal
100 and 1000 is equal to 2 + a decimal, etc.

Therefore, the characteristic of the logarithm of a number with *one* figure to the left of the decimal point is 0; with two figures to the left of the decimal point is 1; with three figures to the left of the decimal point is 2, etc.

From this we may formulate the following rule:

The characteristic of the logarithm of a number greater than 1 is one less than the number of places to the left of the decimal point.

70. From Art. 66 we see that the logarithm of a number between

.1 and .1 is equal to 9 + a decimal — 10
.1 and .01 is equal to 8 + a decimal — 10
.01 and .001 is equal to 7 + a decimal — 10, etc.

Therefore, the characteristic of the logarithm of a decimal with no ciphers between the decimal point and the first significant figure is 9, with a — 10 written after the mantissa; of a decimal with one cipher between the decimal point and the first significant figure is 8, with a — 10 written after the mantissa; of a decimal with two ciphers between the decimal point and the first significant figure is 7, with a — 10 written after the mantissa.

From this we may formulate the following rule:

The characteristic of the logarithm of a number less than 1 is found by subtracting from 9 the number of ciphers between the decimal point and the first significant figure, writing a — 10 after the mantissa.

71. The mantissas of logarithms of numbers are given in

tables of logarithms, and we shall next take up the use of these tables. For the use and application of such tables the student is asked to study pages III, IV and V of Webster Well's six-place logarithmic tables.

72. In the common system, the mantissas of the logarithms of numbers having the same sequence of figures are the same.

Suppose that $\log 2.405 = 0.381115$.

Then
$$\begin{aligned}\log 240.5 &= \log (2.405 \times 100)\\ &= \log 2.405 + \log 100\\ &= 0.381115 + 2\\ &= 2.381115\end{aligned}$$

And
$$\begin{aligned}\log .002405 &= \log (2.405 \times .001)\\ &= \log 2.405 + \log .001\\ &= 0.381115 + 7 - 10\\ &= 7.381115 - 10\end{aligned}$$

In general, if a number be multiplied or divided by an integral power of 10, the resulting number will have the same sequence of figures, and hence the logarithms of the two numbers will have the same mantissas.

73. In the use of logarithms it is necessary to perform the operations of addition, subtraction, multiplication and division. Certain peculiarities in the work will now be explained.

(1) **Addition.** If, after any addition, -10, -20, or -30, etc., is written after the mantissa, and the characteristic is greater than 9, subtract from both parts of the characteristic such a multiple of 10 as will make the characteristic before the mantissa less than 10.

Thus:

$$\begin{array}{rr} & 9.368457 - 10\\ & 8.236425 - 10\\ \hline & 17.604882 - 20\\ \text{or} & 7.604882 - 10\end{array}$$

(2) **Subtraction.** In subtracting a greater logarithm from a less, or in subtracting a negative logarithm from a positive one, the characteristic of the minuend should be increased by 10 (or a multiple of 10), -10 (or $-$ the same multiple of 10) being written after the mantissa.

Thus, to subtract 3.429863 from 2.563298, we arrange the work as follows:

$$\begin{array}{r} 12.563298 - 10 \\ 3.429863 \\ \hline 9.133435 - 10 \end{array}$$

Also in subtracting $9.327891 - 10$ from 4.679231, we arrange the work;

$$\begin{array}{r} 14.679231 - 10 \\ 9.327891 - 10 \\ \hline 5.351340 \end{array}$$

(3) **Multiplication.** After a negative logarithm has been multiplied by an integer, subtract from both parts of the logarithm such a multiple of 10 as will leave -10 written after the mantissa.

Thus, $2 \times (9.526438 - 10) = 19.052876 - 20$, which should be written $9.052876 - 10$.

(4) **Division.** Before dividing a negative logarithm by an integer we add to both parts of the logarithm such a multiple of 10 that after the division a -10 will appear after the mantissa.

Thus,

$$\frac{8.429164 - 10}{3} = \frac{28.429164 - 30}{3}$$
$$= 9.476388 - 10$$

74. Solution of Arithmetical Problems.

(1) Find the value of $1200 \times .02561 \times 76.964$.

$$\log (1200 \times .02561 \times 76.964) = \log 1200 + \log .02561 + \log 76.964.$$

$$\begin{array}{rl} \log\ 1{,}200 = & 3.079181 \\ \log\ .02561 = & 8.408410 - 10 \\ \log\ 76.964 = & 1.886287 \\ \hline \log \text{ Ans.} = & 13.373878 - 10 \\ \text{or} \quad = & 3.373878 \end{array}$$

$\therefore$ Ans. $= 2365.26$

Computation.

Mantissa 7696 = 886265
Tabular diff. = 56 22
.4 886287
22.4

373878
Next less mant. 373831 — No = 2365
Tab. diff. = 184)47. (.255
368
1020
920
1000

(2) Find the value of $\frac{336.842}{7980.2}$.

$$\log \frac{336.842}{7980.2} = \log 336.842 - \log 7980.2$$

log 336.842 = 12.527426 — 10
log 7980.2 = 3.902014
log Ans. = 8.625412 — 10

Ans. = .0422097

Mantissa 3368 = 527372
Tab. diff. = 129 54
.42 527426
54.18

Mantissa 7980 = 902003
11
Tab. diff. = 54 902014
.2
10.8

625412
Next less mant. = 625312 — No = 4220
Tab. diff. = 103)100 (.97
927
730
721

(3) Find the value of $(7.7954)^4$.

$\log (7.7954)^4 = 4 \log 7.7954$ Mantissa 7795 = 891816
$\log 7.7954 = 0.891838$ Tab. diff. = 56 22
4 .4 891838
$\log$ Ans. = 3.567352 22.4
Ans. = 3692.77

567352
Next less mantissa 567262 — No = 3692
Tab. diff. = 117)90. (.77
819
810

(4) Find the value of $\sqrt[3]{.03511}$.

$$\log \sqrt[3]{.03511} = \tfrac{1}{3} \log .03511 = \frac{8.545431 - 10}{3}$$

$$= \frac{28.545431 - 30}{3}$$

$\log$ Ans. = 9.515144 — 10
Ans. = .327449

EXAMPLES FOR PRACTICE.

Calculate by logarithms the value of the following:

(1) $(84.759) \times (2280.76)$. Ans. 193,315.

(2) $(964.38)^{\frac{4}{5}}$. Ans. 244.004.

(3) $4^{\frac{4}{5}} \times 7^{\frac{2}{3}}$. Ans. 13.8289.

(4) $\dfrac{\sqrt{.08}}{(10)^{\frac{3}{4}}}$. Ans. .00710469.

(5) $\dfrac{3.89612 \times .6946}{4694.9 \times .00454}$. Ans. .126965.

THE RIGHT TRIANGLE.

75. In the solution of triangles by logarithms we find it necessary to use the logarithms of the trigonometric functions of angles. In order to find these logarithms we must use the tables of logarithmic sines, cosines, etc., given in our six-place tables. On pages V to X (Well's Six-Place Tables) an explanation of the

table and directions for using it are given. A few examples will be worked out here.

76. (*a*) Find the log sin 82° 17′ 26″.

$$\begin{aligned} \log \sin 82^\circ 17' &= 9.996049 - 10 \\ &\quad\ 7 \\ \log \sin 82^\circ 17' 26'' &= 9.996056 - 10 \end{aligned} \qquad \begin{aligned} D.1'' &= .28 \\ &\quad 26 \\ &\quad 168 \\ &\quad\ 56 \\ &\quad 7.28 \end{aligned}$$

(*b*) Find the log cot 13° 52′ 40″.

$$\begin{aligned} \log \cot 13^\circ 52' &= 0.607553 \\ &\quad\ 361 \\ \log \cot 13^\circ 52' 40'' &= 0.607192 \end{aligned} \qquad \begin{aligned} D.1'' &= 9.03 \\ &\quad 40 \\ &\quad 361.20 \end{aligned}$$

(*c*) Find the angle whose log tan = 9.385490 — 10.

$$\begin{aligned} &9.385337 - 10 \\ \text{Next less log tan} &= 9.385490 - 10 \qquad [\text{angle} = 13^\circ\ 39'] \\ D.1'' &= 9.18)153.\ \ (16.66 \end{aligned}$$

$$\begin{array}{r} 918 \\ \hline 6120 \\ 5508 \\ \hline 612 \end{array}$$

Hence the angle = 13° 39′ 16.7″.

(*d*) Find the angle whose log cos = 9.742512 — 10,

$$\begin{aligned} \text{Next greater log cos} &= 9.742652 - 10 \\ &\quad 9.742512 - 10 \qquad [\text{angle} = 56^\circ\ 26'] \\ D.1'' &= 3.17)140.\ \ (44.16 \end{aligned}$$

$$\begin{array}{r} 1268 \\ \hline 1320 \\ 1268 \\ \hline 520 \\ 317 \\ \hline 2030 \end{array}$$

Hence the angle = 56° 26′ 44.2″.

EXAMPLES FOR PRACTICE.

(1) Find log sin 30° 17′ 8″. Ans. 9.702698 — 10.

(2) Find log cos 47° 35′ 16″. Ans. 9.828956 — 10.

(3) Find log tan 75° 40′ 22″. Ans. 0.592774.

(4) Find log cot 12° 3′ 45″. Ans. 0.670202.

(5) Given log sin $x = 9.983827 - 10$, find x.
Ans. 74° 27′ 37.9″.

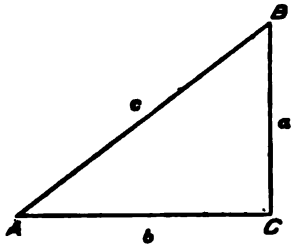
(6) Given log cos $x = 9.877165 - 10$, find x.
Ans. 41° 5′ 35.5″.

(7) Given log tan $x = 0.248200$, find x.
Ans. 60° 32′ 51.2″.

(8) Given log cot $x = 1.169376$, find x.
Ans. 3° 52′ 24.03″.

SOLUTION OF RIGHT TRIANGLES.

77. A right triangle can be completely solved if two sides, or one side and an acute angle, are given, by using the following formulas:



(1) $c^2 = a^2 + b^2$ $\quad A + B = 90°$.

(2) $sin = \dfrac{opposite\ side}{hypotenuse}$. (4) $tan = \dfrac{opposite\ side}{adjacent\ side}$.

(3) $cos = \dfrac{adjacent\ side}{hypotenuse}$. (5) $cot = \dfrac{adjacent\ side}{opposite\ side}$.

Of these formulas select one in which two of the given parts enter, and solve for the third part.

There are five cases which can arise.

Case I. Given the hypotenuse and an acute angle.
Case II. Given a side and the opposite acute angle.
Case III. Given a side and the adjacent acute angle.
Case IV. Given the two sides about the right angle.
Case V. Given the hypotenuse and another side.

In the following computations we shall use certain abbreviations. T. D. = tabular difference; mant. = mantissa; *n. l. m.* = next less mantissa; No. = corresponding number of four places.

78. Case I. Given $\left\{\begin{array}{l} A = 54° 28' \\ c = 125.6 \end{array}\right\}$; find B, a and b.

$$B = 90° - (54° 28') = 35° 32'.$$

Our formulas are, $\sin A = \frac{a}{c}$ (1) and $\cos A = \frac{b}{c}$ (2)

Whence $\log a = \log c + \log \sin A$
and $\log b = \log c + \log \cos A$

$\log c = 2.098990$ 009496
$\log \sin A = 9.910506 - 10$ $n.\,l.\,m. = 009451$ — No. $= 1022$
$\log a = 2.009496$ T. D. $= 425)45.0(.105$
$a = 102.211$
$\log c = 2.098990$ 863298
$\log \cos A = 9.764308 - 10$ $n.\,l.\,m. = 863263$ — No. $= 7299$
$\log b = 1.863298$ T. D. $= 60)35.0(.58$
$\log b = 72.9958$

79. Case II. Given $\left\{\begin{array}{l} A = 67° 9' 23'' \\ a = .235867 \end{array}\right\}$; find B, b and c.

$$B = 90° - (67° 9' 23'') = 22° 50' 37''.$$

Our formulas are, $\tan A = \frac{a}{b}$, and $\sin A = \frac{a}{c}$

Whence $b = \frac{a}{\tan A}$ (1) and $c = \frac{a}{\sin A}$ (2).

From (1),

$\log b = \log a - \log \tan A$
$\log a = 9.372667 - 10$
$\log \tan A = 0.375452$
$\log b = 8.997215 - 10$
$b = .0993607$

mant. 2358 $=$ 372544
T. D. $= 184$ 123
.67 372667
123.28
$\log \tan 67° 9' = 0.375317$
D.1″ $= 5.88$ 135
23 0.375452
135.24

997215
$n.\,l.\,m. = 997212$ — No. $= 9936$
T. D. $= 43)\ 3.\ (.07$

From (2),

$\log c = \log a - \log \sin A$ $\quad \log \sin 67° 9' = 9.964507 - 10$

$\log a = 9.372667 - 10$ $\quad D.1'' = .88$ $\quad 20$

$-\log \sin A = 9.964527 - 10$ $\quad 23$ $\quad 9.964527 - 10$

$\log c = 9.408140 - 10$ $\quad 20.24$

$c = .255941$

408140

$n.l.m. = 408070$ — No. $= 2559$

T. D. $= 170)70.(.41$

80. Case III. Given a side and the adjacent acute angle, we first find the other acute angle and then solve like Case II.

81. Case IV. Given $\left\{\begin{matrix} a = 226.61 \\ b = 543.94 \end{matrix}\right\}$; find A, B and c.

Our formulas are $\tan A = \frac{a}{b}$ (1) and $c = \frac{a}{\sin A}$ (2).

From (1)

$\log \tan A = \log a - \log b$ $\quad$ *Mant.* 2266 $= 355260$

$\log a = 2.355279$ $\quad$ T. D. $= 192$ $\quad 19$

$-\log b = 2.735551$ $\quad .1$ $\quad 355279$

$\log \tan A = 9.619728 - 10$ $\quad 19.2$

$A = 22° 37' 1.3''$ $\quad$ *Mant.* 5439 $= 735519$

$\therefore B = 67° 22' 58.6''$ $\quad$ T. D. $= 80$ $\quad 32$

$.4$ $\quad 735551$

32.0

9.619728

n. l. log sin $= 9.619720$ — angle $= 22° 37'$

D $.1'' = 5.93)\ 8.\ (1.3$

From (2),

$\log c = \log a - \log \sin A$

$\log a = 2.355279$ $\quad \log \sin 22° 37' = 9.584968 - 10$

$-\log \sin A = 9.584975 - 10$ $\quad D.1'' = 5.07$ $\quad 7$

$\log c = 2.770304$ $\quad 1.4$ $\quad 9.584975 - 10$

$c = 89.256$ $\quad 7.098$

770304

$n.l.m. = 770263$ — No. $= 5892$

T. D. $= 73)41.(.56$

82. Case V. Given $\left\{\begin{matrix} c = 47.65 \\ a = 21.84 \end{matrix}\right\}$; find A, B and c.

Our formulas are $\sin A = \frac{a}{c}$ (1), and $b = c \cos A$ (2).

From (1)

$\log \sin A = \log a - \log c$

$\log a = 1.329194$ 9.651131

$\log c = 1.678063$ *n. l.* log sin $= 9.651044$ — angle $= 26° 36'$.

$\log \sin A = \overline{9.651131} - 10$ D.1″ $= \overline{4.22)87.}(20.6$

$A = 26° 36' 20.6''$.

From (2)

$\log b = \log c + \log \cos A$ log cos 26° 36′ = 9.951412

$\log c = 1.678063$ D.1″ = 1.05 22

$\log \cos A = 9.951390 - 10$ 20.6 9.951390

$\log b = \overline{11.629453 - 10}$ 21.630

$\therefore b = 42.6042$ 629453

n. l. m. $= 629410$ — No. $= 4.260$

T. D. $= \overline{102)43.}(.42$

83. A table of natural sines and cosines is sometimes used in solving triangles. We will next solve the example of Case V by this method.

$\sin A = \frac{a}{c} = \frac{21.34}{47.65}$ 44785

$= .44785$ *n. l.* sin $= \overline{44776}$ — angle $= 26° 36'$

$\overline{9.}$

$\therefore A = 26° 36' 20.8''$ D.1″ $= 26$ $\therefore$ D 1″ $= \frac{26}{60}$

correction $= 9 \div \frac{26}{60} = 9 \times \frac{60}{26} = 20.8.$

Since $\cos A = \frac{b}{c}$ cos 26° 36′ = .89415

$\therefore b = c \cos A$ D.1′ = 13 · 5

.89410

$= 47.65 \times .89410$ cor. $= \frac{13}{60} \times 20.8 = 4.51$

$= 42.603865$

$\therefore b = 42.6039$

EXAMPLES FOR PRACTICE.

Solve the following right triangles:

(1) Given A $= 15°$, $c = 7$. Ans. $a = 1.8117, b = 6.7615$.

(2) Given A $= 38°$, $a = 8.09$.

Ans. $b = 10.3547$, $c = 13.1404$.

(3) Given A $= 72°\ 52'$, $b = 6.306$.

Ans. $a = 20455.6$, $c = 21405.6$.

(4) Given $a = 529.3$, $c = 902.7$.

Ans. A $= 35°\ 53'\ 55.2''$, $b = 731.237$.

(5) Given $a = 3.414$, $b = 2.875$.

Ans. A $= 49°\ 53'\ 55''$, $c = 4.46330$.

(6) Given A $= 26°\ 12'\ 24''$, $c = 469422.7$.

Ans. $a = 207302, b = 421{,}170$.

(7) Given B $= 82°\ 6'\ 18''$, $a = 89.32$.

Ans. $b = 644.109$, $c = 650.272$.

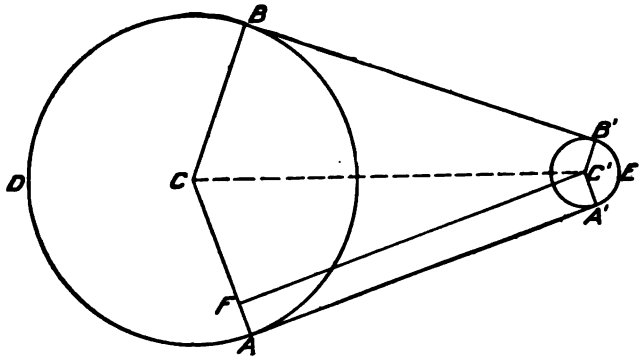


Fig. 17.

84. Problem. Find the length of a belt which passes around two pulleys, the radii of which are 6 feet and 1 foot respectively, and the distance between centers 13 feet.

In Fig. 17, let C C′ = 13 feet, C A = 6 feet and C′ A′ = 1 foot. Draw C′ F parallel to A A′. Then C F = 6 — 1 = 5 feet. In the right triangle C F C′, C C′ = 13, and C F = 5.

Therefore, $C'F = \sqrt{\overline{CC'}^2 - \overline{CF}^2}$

$= \sqrt{169 - 25} = \sqrt{144} = 12.$

We will now find the magnitude of angle F C C′ in triangle F C C′.

$$\sin \text{F C C}' = \tfrac{12}{13}$$
$$\log \sin \text{F C C}' = \log 12 - \log 13$$
$$= 1.079181 - 1.113943$$
$$= 9.965238 - 10$$

Therefore, angle F C C′ = 67° 22′ 49.4″

And 2 (F C C′) or A C B = 134° 45′ 37.6″

Subtracting this last result from 360°, we get the angle subtended by the arc A D B:

$$360° - 134° 45' 37.6'' = 225° 14' 22.4''$$

By Geometry, the arcs of circles are proportional to the subtended angles.

Hence, $$\frac{arc\ \text{A D B}}{\text{circumference of the circle}} = \frac{225° 14' 22.4''}{360°}$$

Or $$\frac{arc\ \text{A D B}}{2\pi R} = \frac{225.2396}{360}$$

Letting $\pi = 3.1416$,

$$arc\ \text{A D B} = 37.6992 \times \frac{22.52396}{36}$$

Solve this by logarithms.

$$\log arc\ \text{A D B} = \log 37.6992 + \log 22.52396 - \log 36$$
$$\log 37.6992 = 1.576332$$
$$\log 22.52396 = 1.352645$$
$$2.928977$$
$$\log - 36 = 1.556303$$
$$\log arc\ \text{A D B} = 1.372674$$
$$arc\ \text{A D B} = 23.587$$

Similarly, $$arc\ \frac{\text{A}' \text{E B}'}{\text{circum. small circle}} = \frac{225.2396°}{360°}$$

$$\therefore arc\ \text{A}' \text{E B}' = 6.2832 \times \frac{22.52396}{36}$$

$$\log 6.2832 = 0.798185$$
$$\log 22.52396 = 1.352645$$
$$2.150830$$
$$\log - 36 = 1.556303$$
$$.594527$$
$$\therefore arc\ \text{A}' \text{E B}' = 3.93122$$

$$\text{Length of belt} = A\,A' + A'\,E\,B' + B\,B' + B\,D\,A$$
$$= 12 + 3.931 + 12 + 23.587$$
$$= 51.518$$

85. In Mechanics we are required at times to solve right triangles.

Problem. Suppose that the rod A C, which sustains a pull of 500 pounds, is to be replaced by the two rods A B and A D, each making an angle of 15° with A C. How great a pull must each exert?

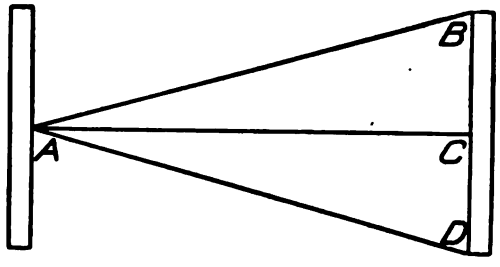
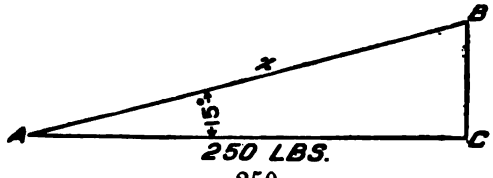


Fig. 18.

From Mechanics we know that the component of A B along A C plus the component of A D along A C must equal the force along A C or 500 pounds. Therefore the component of A B along A C must equal 250 pounds. Then we may represent the forces thus:



Then, $$\cos 15° = \frac{250}{x}$$

Or, $$x = \frac{250}{\cos 15°}$$

$$\log x = \log 250 - \log \cos 15°$$
$$\log 250 = 2.397940$$
$$-\log \cos 15° = 9.984944 - 10$$
$$\log x = \overline{2.412996}$$
$$x = 258.82 \text{ the force each must bear.}$$

SOLUTION OF OBLIQUE TRIANGLES.

86. There are four cases of oblique triangles.

Case I. Given one side and two angles.
Case II. Given two sides and the angle opposite one of them.
Case III. Given two sides and the included angle.
Case IV. Given the three sides.

87. **Cases I and II** can be most easily solved by using the following formula:

The sides of a triangle are proportional to the sines of the opposite angles.

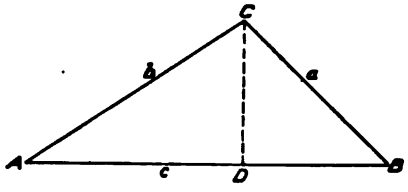


Fig. 19.

Let A B C be the given triangle.

In each figure the perpendicular, C D, comes in two right triangles.

In one $\quad C\,D = b \sin A$

In the other $\quad C\,D = a \sin B$

Therefore $\quad a \sin B = b \sin A$

Or $$\frac{a}{b} = \frac{\sin A}{\sin B} \qquad [38]$$

Similarly $$\frac{b}{c} = \frac{\sin B}{\sin C} \qquad [39]$$

$$\frac{a}{c} = \frac{\sin A}{\sin C} \qquad [40]$$

88. **Case I.** Given $A = 40° 20'$, $B = 57° 30'$, $b = 76$

Then $$C = 180° - (A + B)$$
$$= 82° 10'$$

We have from Art. 87

$$\frac{a}{b} = \frac{\sin A}{\sin B} \text{ (1) and } \frac{c}{b} = \frac{\sin C}{\sin B} \text{ (2)}$$

From (1)		From (2)	
$\log a = \log b + \log \sin A - \log \sin B$		$\log c = \log b + \log \sin c - \log \sin B$	
$\log b =$	1.880814	$\log b =$	1.880814
$\log \sin A =$	9.811061	$\log \sin c =$	9.995928
$=$	1.691875	$=$	1.876742
$\log \sin B =$	9.926029	$\log \sin B =$	9.926029
$\log a =$	1.765846	$\log c =$	1.950713
$a =$	58.3238	$C =$	89.2714

89. Case II.

$$\text{Given} \begin{cases} b = 3.3725 \\ a = 2.0986 \\ A = 56^\circ\ 3' \end{cases}$$

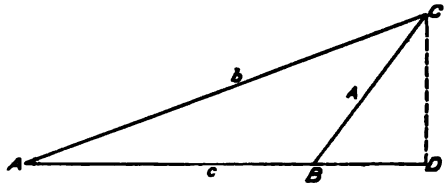


Fig. 20.

Our formula is $$\frac{b}{a} = \frac{\sin B}{\sin A}$$

Therefore $$\sin B = \frac{b \sin A}{a}$$

$$\log \sin B = \log b + \log \sin A - \log a$$

$\log b =$	0.321929
$\log \sin A =$	9.918830 — 10
	10.240759
$\log a =$	0.527952
$\log \sin B =$	9.712807 — 10

$$\therefore B = 31^\circ\ 4'\ 36.6''$$

$$\text{and } A + B = 87^\circ\ 7'\ 36.6''$$

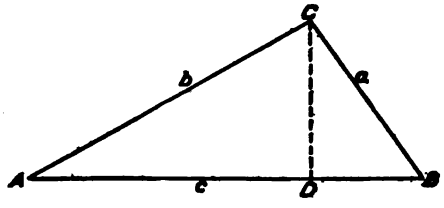
When $C = 180^\circ - (A + B) = 92^\circ\ 52'\ 23.4''$

To find c, use $$\frac{c}{a} = \frac{\sin C}{\sin A}$$

Whence, $$c = \frac{a \sin C}{\sin A}$$

$$\begin{aligned}
\log c &= \log a + \log \sin C - \log \sin A \\
\log a &= \ \ 0.527952 \\
\log \sin C &= \ \ 9.999454 - 10 \\
&\quad\ \ 10.527406 - 10 \\
\log \sin A &= \ \ 9.918830 - 10 \\
\log c &= \ \ .608576 \\
c &= \ \ 4.06047
\end{aligned}$$

90. Cases III and IV are usually solved by first obtaining formulas connecting the given parts and a required part, but as we have not deduced these formulas we will solve these cases by breaking up the oblique triangles into right triangles.



91. Case III. $b = 35.31$; $c = 42.18$; $A = 31° 26'$.

From the right triangles we have the following:

$$\begin{aligned}
&(1) \quad C\,D = b \sin A \\
&(2) \quad A\,D = b \cos A \\
&(3) \quad D\,B = c - A\,D \\
&(4) \quad \tan B = \frac{C\,D}{D\,B} \\
&(5) \quad a = \frac{C\,D}{\sin B} \\
&(6) \quad C = 180° - (A + B)
\end{aligned}$$

From (1)

$$\begin{aligned}
\log C\,D &= \log b + \log \sin A \\
\log b &= 1.547898 \\
\log \sin A &= 9.717259 - 10 \\
\log C\,D &= 1.265157
\end{aligned}$$

From (2)

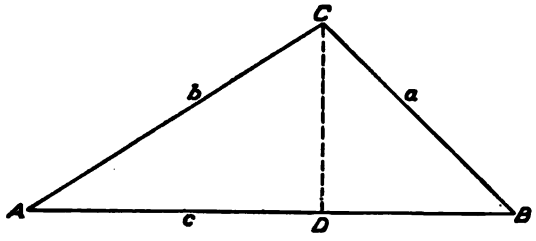
$$\begin{aligned}
\log A\,D &= \log b + \log \cos A \\
\log \cos A &= \ \ 9.931075 - 10 \\
\log b &= \ \ 1.547898 \\
\log A\,D &= \ \ 1.478973 \\
A\,D &= 30.1282 \\
\therefore D\,B &= 42.18 - 30.1282 \\
&= 12.0518
\end{aligned}$$

From (4)	From (5)
$\log \tan B = \log C D - \log D B$	$\log a = \log C D - \log \sin B$
$\log C D = 1.265157$	$\log C D = 1.265157$
$-\log D B = 1.081052$	$-\log \sin B = 9.922584$
$\log \tan B = \overline{0.184105}$	$\log a = \overline{1.342573}$
$B = 56° 47' 46.1''$	$a = 22.0076$
$C = 180° - 88° 13' 46.1''$	$= 91° 46' 13.9''$

92. Case IV. Given a, b, c.

$$c = A D + D B \quad (1)$$

In right triangles A D C and B D C,



$$b^2 = \overline{A D}^2 + \overline{D C}^2$$

$$a^2 = \overline{B D}^2 + \overline{D C}^2$$

By subtracting $\quad b^2 - a^2 = \overline{A D}^2 - \overline{B D}^2$

Factoring $\quad (b+a)(b-a) = (A D + D B)(A D - D B)$

$$= C (A D - D B)$$

Hence $\quad A D - D B = \dfrac{(b+a)(b-a)}{c} \quad (2)$

By taking half the sum of A D + D B and A D — D B we find a value for A D. By taking half the difference of these same expressions we find a value for D B.

Then, to find the angles we have

$$\cos A = \frac{A D}{b}, \quad \cos B = \frac{D B}{a}$$

and $\quad C = 180° - (A + B)$

Example. Let $a = 14$; $b = 16$; $c = 20$.

Then $\quad A D + D B = 20$

And from (2) $AD - DB = \frac{30 \times 2}{20} = 3$

Add $2AD = 23$

$AD = 11.5$

Subtract $2DB = 17$

$DB = 8.5$

$\cos A = \frac{11.5}{16} = 0.71875$

$\therefore A = 44° 2' 54.3''$

$\cos B = \frac{8.5}{14} = .60714$

$B = 52° 37'$

and $C = 83° 20' 5.7''$

EXAMPLES FOR PRACTICE.

Solve the triangles:

(1) Given $a = 180$, $A = 38°$, $B = 75° 43'$.
(2) Given $c = 24.637$, $A = 83° 39'$, $B = 38° 56'$.
(3) Given $a = 5.98$, $b = 3.59$, $A = 63° 50'$.
(4) Given $b = .41782$, $c = .38279$, $C = 29° 46'$.
(5) Given $a = 886$, $b = 747$, $C = 71° 54'$.
(6) Given $b = 4.102$, $c = 4549$, $A = 62° 9' 38''$.
(7) Given $a = 5$, $b = 7$, $c = 6$.
(8) Given $a = 10$, $b = 9$, $c = 8$.

ANSWERS.

(1) $b = 283.331$, $c = 267.677$.
(2) $a = 29.0595$, $b = 18.3742$.
(3) $B = 32° 36' 9.4''$, $c = 6.62085$.
(4) $B_1 = 32° 48' 47.4''$, $a_1 = .684403$.
and $B_2 = 147° 11' 12.6''$, $a_2 = .0409772$.
(5) $A = 60° 44' 39.5''$, $c = 965.282$.
(6) $C = 63° 49' 9.3''$, $a = 4.48237$.
(7) $A = 44° 24' 54.8''$, $B = 78° 27' 47.0''$, $C = 57° 7' 17.6''$.
(8) $A = 71° 47' 24.4''$, $B = 58° 45' 5.4''$, $C = 49° 27' 30.0''$.

AREAS OF TRIANGLES.

93. The area of a triangle is one-half the product of its base by its altitude. Hence, if we denote the area of the triangle by K, the formula for the area of the right triangle in Art. 77 is

$$2\,K = a\,b.$$

This formula used in connection with those in Art. 77 will enable us to find the area of any right triangle.

94. In oblique triangles we may have three cases.

Case I. Given two sides and the included angle.

In Figs. 19 and 20, Art. 87,

$$2\,K = c \times C\,D$$

But $$C\,D = b \sin A.$$

Hence $$2\,K = b\,c \sin A.$$

Similarly $$2\,K = a\,b \sin C.$$

$$2\,K = c\,a \sin B.$$

Case II. Given one side and the three angles.

From Case I,

$$2\,K = b\,c \sin A. \qquad (1)$$

And from Art. 87,

$$\frac{b}{c} = \frac{\sin B}{\sin C}.$$

$$\therefore\ b = \frac{c \sin B}{\sin C}.$$

Substituting in (1) for b the value just found,

$$2\,K = \frac{c^2 \sin A \sin B}{\sin C}.$$

Similarly $$2\,K = \frac{a^2 \sin B \sin C}{\sin A}$$

And $$2\,K = \frac{b^2 \sin C \sin A}{\sin B}.$$

Case III. Given the three sides.

Unless we use the formula [37] deduced in Art. 54, we must break up the oblique triangle into right triangles according to the method of Art. 92. It is better to use formula [37].

EXAMPLES FOR PRACTICE.

1. Find the 9th power of 1.179.

Ans. 4.401765+

2. Find the 7th root of —17.

Ans. —1.49891+

Find remaining parts in the following right triangles:

3. Given $b = 217$, $c = 915$.

Ans. $\begin{cases} a = 888.896 \\ A = 76° 16' 52'' \\ B = 13° 43' 8'' \end{cases}$

4. Given $a = 174$, $b = 1927$.

Ans. $\begin{cases} A = 5° 9' 34'' \\ B = 84° 50' 26'' \\ c = 1934.89 \end{cases}$

5. Given $a = 17.94$, $A = 15° 39'$.

Ans. $\begin{cases} b = 64.038 \\ c = 66.503 \end{cases}$

Find remaining parts in the following oblique triangles:

6. Given $A = 45° 4'$, $B = 75° 35$, $c = 457$.

Ans. $\begin{cases} C = 59° 21' \\ a = 376.06 \\ b = 514.48 \end{cases}$

7. Given $a = 57$, $b = 89$, $C = 75° 4'$.

Ans. $\begin{cases} A = 36° 32' 37'' \\ B = 68° 23' 23'' \\ c = 92.495 \end{cases}$

8. Given $c = 919$, $A = 37° 37'$.

Ans. $\begin{cases} a = 560.94 \\ b = 727.95 \end{cases}$

9. Given $b = 45.3$, $A = 34° 23'$.

Ans. $\begin{cases} a = 30.998 \\ c = 54.890 \end{cases}$

SECTION 3.

NAVIGATION.

Although elementary in character, this book is complete enough to enable the student to solve the ordinary problems of navigation. It is, therefore, essential that the student, if he wishes a mate's or a master's paper, thoroughly understand the contents. As the student has already studied arithmetic, geometry and trigonometry, these subjects will not be repeated except in a few cases. This is not a book on seamanship, but is intended for seamen who know little of the subject of navigation. Ignorance of navigation often blocks the way to a position of command.

As most navigators have at hand several reference books, such as "American Practical Navigator" by Bowditch and "Wrinkles in Practical Navigation" by Lecky, these books have been referred to for tables, etc.

DEFINITIONS.

A sphere is a solid body, every part of whose surface is equidistant from a fixed point within, called the center.

A diameter of a sphere is any straight line passing through the center and extending in both directions to the surface.

A great circle is any circle whose plane passes through the center of the sphere. The center of the sphere is the common center of all great circles.

The shortest distance between any two points on a sphere is the *arc* of a great circle passing through those points.

A *small circle* is any circle whose plane does not pass through the center of the sphere.

A spherical angle is the angle or inclination of two great circles of a sphere which meet one another.

A spherical triangle is the geometrical figure formed on the surface of a sphere by the intersections of arcs of three great circles.

The earth, generally speaking, is a sphere, and revolves

about one of its diameters, called the axis, once in every twenty-four hours. The extremities of this axis are called poles, one the north and the other the south pole; in Fig. 1 N S is the earth's axis, N being the north and S the south pole.

Equator. This is an imaginary great circle dividing the surface of the earth into two parts, called hemispheres, the north and south. The equator, which is also commonly called the "Line," is represented in Fig. 1 by Q T.

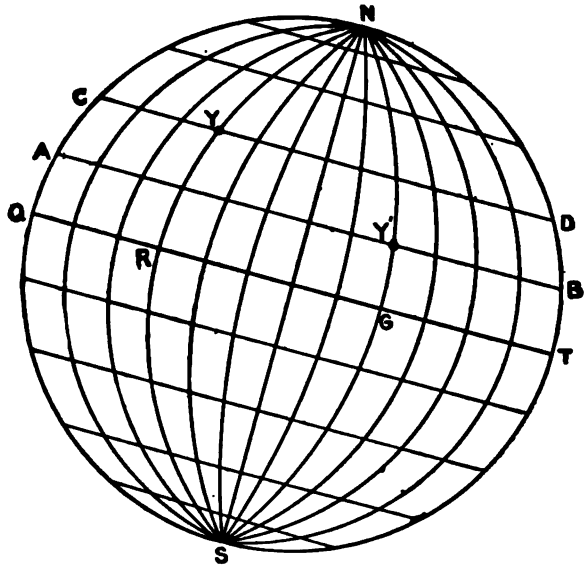


Fig. 1.

Meridians. These are great circles on the earth's surface drawn through both poles and at right angles to the equator. They are called meridians because they mark the places which have noon at the same time. For instance, N Q S of Fig. 1 is a meridian, also N R S, and the other curves drawn through N and S. All places on N R S have noon at the same time. Any two places which lie exactly north or south of each other are said to be on the same meridian. One particular meridian, the one passing through the observatory at Greenwich, England, is taken by the English and American nations to be the first, or prime, meridian.

Longityde. The longitude of a place is the distance of its meridian, or the meridian passing through it from the prime meridian; this distance is measured on the equator in degrees, minutes and seconds. Another definition is: longitude is the angle at the pole between the two aforesaid meridians. In Fig. 1 let Y be a place on the meridian N R S, and let N G S be the prime, or Greenwich, meridian; then G R is the longitude of Y and is west of the prime meridian. All places lying between the prime meridian and the 180° meridian in a westerly direction from the prime meridian are in west longitude, and in east longitude when lying between the same meridians in an easterly direction from Greenwich. The 180° meridian is just halfway around the world from the Greenwich meridian.

Parallels. Parallels of latitude are small circles parallel to the equator, as A B, C D, etc. (Fig. 1). All places on parallel A B, for instance, are exactly true east and west of each other.

Latitude. This is the name given to the distance of a place north or south of the equator, expressed in degrees, minutes and seconds measured on a meridian. In Fig. 1, Y is on the parallel C D, and its latitude, expressed in degrees, minutes and seconds, is the arc R Y of the meridian N R S. We can now see how latitude and longitude determine the position of any ship or other object on the surface of the earth, for longitude fixes its distance east or west of Greenwich and latitude its distance north or south of the equator. The equator, meridians and parallels of latitude are of course purely imaginary lines and are used simply as a means of locating position.

The Difference of Longitude between two positions is the distance measured on the equator between their meridians.

The Difference of Latitude is the distance between the two parallels of latitude on which the two positions are situated measured on a meridian. The difference of longitude between Y′ and Y (Fig. 1) is G R, the difference of latitude is B D, A C, or the distance between the parallels A B and C D measured on any meridian.

Departure is the distance in nautical miles made by a ship due east or west, according as it is sailing toward the east or the west, and is commonly called "easting" or "westing." Easting

or westing is expressed in *nautical miles*, and is unlike longitude in this respect. It might seem at first sight that departure and longitude were one and the same thing, but as already stated, longitude is expressed in degrees, etc., while departure is expressed in nautical miles. The difference is this: the distance of length of a degree of longitude grows less as we go north or south from the equator, while the distance of a nautical mile is the same everywhere. The distance of one nautical mile on the equator is equal to one minute of longitude; north or south of the equator the distance of one mile is greater than one minute of longitude. It will therefore be plain that theoretically it is better to make easting or westing in as high latitudes as possible, for the higher the latitude the shorter the distance necessary to sail in order to change the same distance of longitude.

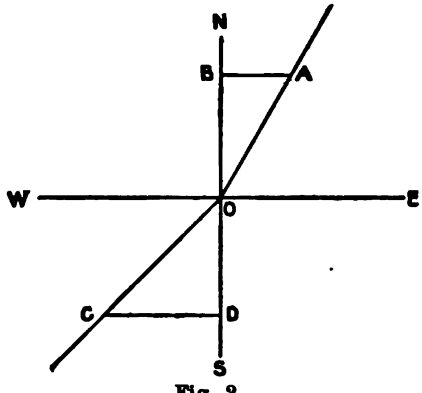


Fig. 2.

The course steered is the angle between the meridian and the ship's head. The course made good is the angle between the meridian and the real track of the ship.

In Fig. 2 let N S be a true north and south line, or meridian, and W E an east and west line, or parallel. Now let O A be the direction along which the ship sails in going from O to A; then O A is the distance sailed, O B the difference of latitude, B A the departure, and the angle B O A the course; similarly, O C is the distance sailed, D O the difference of latitude, C D departure, and D O C the course.

Nautical Miles, or Knots. A great circle of the earth is divided into 360 degrees, each degree into 60 minutes, and each minute into 60 seconds. Each minute is 6,080 feet (nearly) in length, or one nautical mile, the statute mile being only 5,280 feet.

Rhumb Line. This is the line on which the ship sails as long as her course is not altered; it forms a curve upon the earth's surface which meets every meridian at the same angle. It coin-

cides with a meridian when sailing true north or south, and with a parallel when sailing true east or west. The angle between the rhumb line and the meridian is the course.

THE COMPASS.

Before taking up the actual problem of navigating the ship we must consider the compass and its behavior, together with its corrections. A short description of this most important navigating instrument will be sufficient; its action and errors, however,

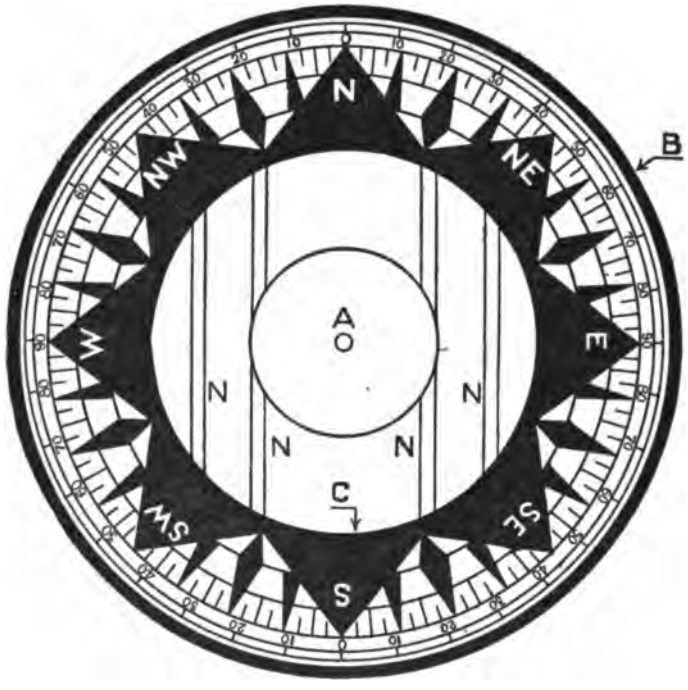


Fig. 3.

will be explained at greater length. The mariner's compass consists of a circular brass card C (Fig. 3), covered with white enamel, on which are marked the points and degrees as shown. The figure shows marks at 10 degrees, but a regular compass has the one degree marks as well. The smaller compasses often do not have degree marks, but simply the points. B represents the edge of the compass bowl, which swings in gimbals and is carried in a binnacle. A is a spheroidal air vessel, N are the bundles of magnetic

needles. The card, needles and air vessel are fastened rigidly together and rest on a pivot at the center of A, the bowl being filled with alcohol and covered by glass.

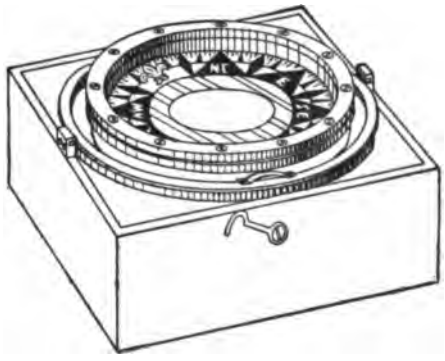
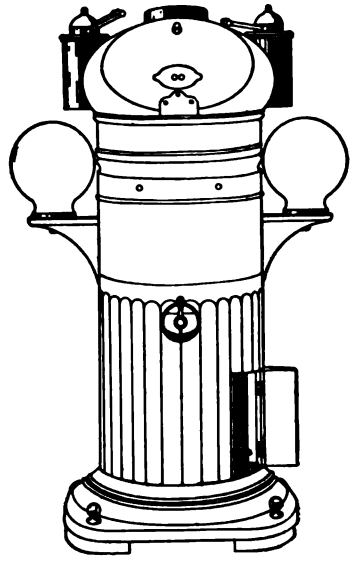


Fig. 4.



BINNACLE.

Fig. 4 shows a compass in a box, and Fig. 5 an improved form of compass card.

The circular card is divided into degrees and points to represent every different point of the horizon. Every circle being divided into 360 degrees, and there being 32 points to the compass card, the angle between the single points will be $\frac{1}{32}$ of 360°, or 11° 15′. The points of the compass are as follows, commencing at the north point and going around right handed:

N,	N by E,	NNE,	NE by N,	NE,	NE by E,	ENE,	E by N,
E,	E by S,	ESE,	SE by E,	SE,	SE by S,	SSE,	S by E,
S,	S by W,	SSW,	SW by S,	SW,	SW by W,	WSW,	W by S,
W,	W by N,	WNW,	NW by W,	NW,	NW by N,	NNW,	N by W.

Each point is further divided into quarter points; for instance, $\frac{1}{2}$ a point E of N is called N $\frac{1}{2}$ E, a quarterway between SW and SW by S is SW $\frac{1}{4}$ S, and so on. These points, or "boxing the compass," as it is called, should be well learned.

A course is spoken of commonly as a 1, 2, 3, or 4 point course, as the case may be; for instance, N by W is a 1 point course, NNE, SSE, SSW, NNW are 2 point courses, NE, SE, SW and

NW are 4 point courses. It will be seen at a glance that the notation is not difficult.

Naming the Course. As already stated, the course is the angle reckoned from the N or S towards E or W; for instance, if a ship is steering by compass N by E, the course is a one-point course, and is one-point to the right of N. E by S is not a nine-point course, but a seven-point course, seven points to the left of S.

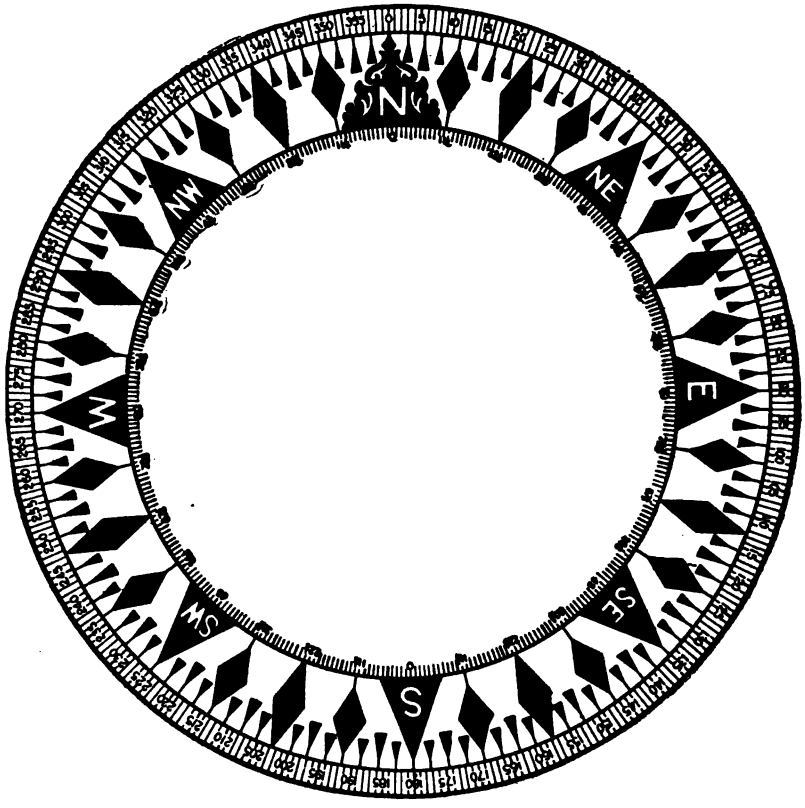


Fig. 5.

E and W are eight-point courses, in reality, although never spoken of as such.

On the inside of the compass bowl is a vertical black line in the direction of the ship's bow; this line, called the lubber line, shows the helmsman the direction of the ship's head. If, for example, he is told to steer SSE, he puts the helm over so as to bring the lubber line opposite SSE on the compass card.

Magnetism. Just what this is, is not known, but through careful study certain laws have been formulated which govern all magnets. There are two kinds of magnets, natural and artificial. The earth is a natural magnet, the compass needle an artificial. The ends of a magnet are called *poles*, and the magnetism at one pole is of a different kind from that of the other, inasmuch as its properties are different. If we call one pole north and the other south, the north pole of one magnet when brought up to the north pole of another magnet will repel, while if placed near the south pole it will attract; thus, like poles repel and unlike poles attract. This law accounts for the action of the compass needle; as the earth is a magnet, one of its magnetic poles will attract the pole of another magnet. The magnetic pole of the earth in the northern hemisphere is called the north magnetic pole, and while the north point of the compass card or magnet is attracted toward the north magnetic pole, yet the magnetisms are opposite. If we consider the north magnetic pole of the earth to be north magnetism, then the north point of the compass needle is, according to the law stated, south magnetism. This may be a bit confusing, but if we call the magnetism at the end of a magnet which is attracted by the earth's north pole, north-seeking magnetism, and that of the opposite pole south-seeking magnetism, the confusion may be overcome. To sum up, we may say that the north point of the compass card has north-seeking magnetism and the south point south-seeking magnetism. The north-seeking end of a magnet is often painted red and the south-seeking end blue.

The north magnetic pole of the earth is not situated at the real north pole, but is approximately 70° 5′ North Latitude, 96° 46′ West Longitude.

Dip of Needle. A magnetic needle balanced on its center of gravity, free to move in any direction, does not remain horizontal, but its north point in the north hemisphere points downward, or dips. At the magnetic poles the needle would be vertical, while at the magnetic equator the needle would be horizontal.

Variation. The compass needle very seldom points true north, but to the magnetic north. The angle between the needle and the true north and south line, or meridian, when free from any local attraction, is called the variation at the place. The variation

is always marked on charts, because the variation is different for different places, being E at some and W at others. By westerly variation we mean that the north point of the needle is pulled to the left of the true north, and easterly when pulled to the right. Suppose we are at a place where the variation is 22° 30′ W and that the ship's head is pointing true north, the north point of the compass will be drawn, owing to the variation 22° 30′, or 2 points, to the left of the true north, and would thus indicate that the ship was headed NNE. We know, however, that she is headed true north, hence to correct for variation we have the following rule: *When the variation is westerly the true course is equal to as many points to the left of the compass course as the variation, and when easterly the correction is applied to the right; looking in each case from the center of the card toward the point to be corrected.*

The following examples should be carefully studied, as this is a matter of importance:

Ex. 1. Compass course S $\frac{3}{4}$ W, variation $2\frac{1}{4}$ points E. True course $2\frac{1}{4}$ points to the right of S $\frac{3}{4}$ W or SW by S.

Ex. 2. Course by compass N by E, variation 2 points W. True course is 2 points to the left of N by E or N by W.

Ex. 3. Course or bearing N 84° E, variation 19° W. True bearing N 65° E.

Now if we have the true course and wish to find the compass course we apply the variation in the opposite way. Suppose we wish to steer a ship so as to reach a certain place which lies SW true from our starting-point, the variation being 2 points E, therefore we cannot steer by compass SW, but must steer by compass SSW, which is in this case the compass course. Practice in correcting courses is essential, and the student should keep at it until he thoroughly understands the process.

Deviation. The iron in the hull of a ship, the cargo in the hold, tanks, etc., and the subpermanent magnetism received while the vessel is being built, affect the compass. These elements cause an error called deviation. Whatever error arises from objects within a ship is called deviation; whatever by objects outside the ship is called local attraction. Closely connected with local attraction and deviation, we have what is known as perma-

nent and subpermanent magnetism. All magnets have the power of imparting magnetism to certain other substances, when brought within their influence. Soft iron in any form is most susceptible to this power; and the earth being a great magnet induces, as it is called, magnetism in any iron body lying in the direction of its line of magnetic force or the magnetic meridian. If the iron body be kept for a long time in one position, and especially if it be subjected to hammering, the magnetism it is receiving from the earth becomes of a lasting character, and remains, to a certain extent, after the body is removed; as the poles of magnets induce magnetism of an opposite name in soft iron bodies under its influence, the part of the iron body which is pointed toward the north magnetic pole of the earth will receive red magnetism and the opposite part blue magnetism. It will now be understood why the direction of a ship's head during building, and the hammering to which she is subjected, cause a ship to acquire a large amount of induced magnetism, and this once hammered in, though it will be diminished by change of position, knocking about by waves, etc., will never, while the iron remains in the form of the ship, be entirely eradicated. This is subpermanent magnetism, so called. Therefore we see that another disturbing element is at work to change the direction of the compass needle or to produce error; this element being, as stated, iron used in the ship's construction, her equipment, or carried as cargo. These disturbing forces, permanent and subpermanent magnetism, have been named (according to the effects they produce on the compass needle) *semicircular* and *quadrantal deviation.*

Semicircular deviation is caused by subpermanent magnetism and induced magnetism in vertical iron, such as masts, funnels, etc. Its greatest effects are produced when the ship's head is on the cardinal points; it is called semicircular because it produces easterly deviation in one semicircle and westerly in the other.

Quadrantal deviation is caused by the induced magnetism in horizontal iron; it produces easterly deviation in one quadrant, westerly in the next, and so on. To sum up, vertical iron is semicircular in action, horizontal iron is quadrantal in its action.

The *heeling* of the ship alters the deviation by changing vertical and horizontal iron to inclined. It might seem that this

was of little consequence; but such is not the case, and an intelligent navigator should understand this heeling error clearly, and allow for it. The effect of heeling error increases as the distance from the magnetic equator increases, at which place it is nothing. The following should be taken account of: In the northern latitudes on northerly courses heeling error may be expected to place the ship to windward and on southerly courses to leeward of her supposed position.

Leeway. The action of the wind, the trend of the sea, etc., tend to drive a ship to leeward, so that she does not move exactly in the direction toward which she is heading, but is driven out of her course somewhat. This amount is called leeway, and may be defined as the angle between the course which she is actually going and that which she is trying to go. The estimation of the amount of leeway is one of the tricks of the mariner's trade, so to speak; it may be judged by looking at the wake.

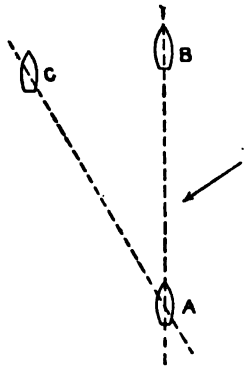


Fig. 6.

Referring to Fig. 6, let a ship be headed or steering as indicated by dotted line A B, but owing to the wind the ship actually goes in the direction A C; the angle C A B being the leeway.

In correcting for *leeway* when on the *starboard tack* it must be allowed to the left, the same as westerly variation, etc.; on the *port tack* it must be reckoned to the right, the same as easterly variation.

CORRECTION OF COURSES.

From the foregoing discussion we know that the compass does not show the direction in which the ship is truly headed. It is plain that the navigator, knowing what course he has been steering by compass, must, in order to locate the ship's position accurately, find out the course he has actually made. Conversely, knowing the true course he must take in order to reach a certain place, he must know what the compass course shall be in order to make the place in question. This will be illustrated by an example, after stating the rules for correcting courses. It is of such

importance that the student should study it carefully so as to leave no doubt in his mind as to how to correct courses.

To find true course from compass course. Put down your compass course, calling it so many points or degrees, then allow easterly variation to the right and westerly to the left; allow for deviation in the same way; leeway on the starboard tack is the same as westerly variation, hence is allowed to the left or away from the wind, and on the port tack the same as easterly variation.

Every master should know how to find the deviations of his compass. Now for a practical example of the use of the above-stated rules for correcting courses: Fig. 7 is a facsimile of a regular U. S. chart of a portion of the Atlantic coast, and is adapted from Atlantic coast chart No. 1,000, from Cape Sable to Hatteras. Let A B C D E F, etc., be the track of a vessel bound into Boston; it will be assumed that from A to F the wind has held N, ½ E, about a six-knot breeze, and that about the time the ship reached position F it hauled to NNE ½ E; this would give nearly a fair wind for running to Boston, and it is now necessary to find what course must be steered to make Boston Lightship. We find by use of the parallel rule (referring it to the true compass card in the lower right-hand corner of the chart), or by the horn protractor (for description see page 28), that the Lightship bears W by N ½ N, which is our true course or course to be made good. Let the leeway be estimated at 1 point. The ship is on the starboard tack; we will assume that there is no bad steerage. Let the deviation be 1 point E; the variation given on above-mentioned compass card is 13½° W, which for simplifying we will assume as being 1 point W, to find compass course or course to be steered. True course = W by N ½ N. The variation being equal and opposite the deviation, they counteract each other, leaving 1 point leeway to be corrected or allowed for, which to find the compass course is allowed to the right or toward the wind, which makes the compass course WNW ½ N.

Now take an exact case

True course W by N ½ N = 73° 7½′ W
Deviation 10° E
Variation 13½° W
Leeway 11¼ Starboard tack same as W variation.
14¼° W

This is now allowed to the right of the true course N 73° $7\frac{1}{2}'$ W, and gives N 58° $22\frac{1}{2}'$ W, or NW by W, which is given to the helmsman.

Now suppose we had been sailing by compass for a certain number of hours NW by W, and that the leeway, variation and deviation were the same as in the above case; then in order to find the true course made good by the ship, we reverse the preceding methods and get as the true course W by N $\frac{1}{2}$ N, which is necessary if the ship's track is to be marked upon the chart.

CHARTS.

A chart is a representation on a flat surface, such as paper, of a portion of the surface of the earth. Charts are made by several different projections. Harbors and small sections are made on what is called the Polyconic projection, while for great circle sailing (to be explained later) the charts used are on the Gnmonic projection. For general navigation, charts constructed on Mercator's projection are used almost entirely. A short description of Mercator's chart will be given, and a description of the other projections may be found in most books on Navigation.

Mercator's Chart. On this chart all the meridians and parallels are straight lines; the degrees of longitude are all equidistant; the degrees of latitude are increased in length from the equator to the pole in the same proportion as the degrees of longitude actually decrease on the earth's surface.

Use of Chart. The chart is spread out on a chart table, with the south end nearest the navigator. General charts for ocean sailing, and covering considerable extent, such as a chart of the North Atlantic, have, in addition to the true meridians, the magnetic meridians or lines of equal variation, also small compass cards (see Fig. 7). Under the title of the chart sufficient explanation is given whereby the chart may be read.

Parallels and meridians are at right angles to each other.

Scale of Latitude and Distance. The two meridians which bound the chart, as 67° and 75°, Fig. 7, are called graduated meridians, because they are marked to degrees and minutes. On these meridians *latitude* is measured from the equator toward the pole in degrees and minutes, according to the scale of the sub-

divisions. *The distance between two places is also measured on a graduated meridian; a minute being equal to one mile.*

Scale of Longitude. The two parallels which bound the top and bottom of the chart are called graduated parallels, 44° and 40°; they are also marked to degrees (and minutes, if the scale is large enough). *Longitude* is measured on these parallels and nothing more; the *distance between two places cannot be measured on the parallels.* On Mercator's chart the track of the ship, or rhumb line, is straight.

To Find Latitude of a Place on the Chart. With the dividers (for description see page 28) take the distance of the place from the nearest parallel, and refer this distance to one of the graduated meridians, placing one point of the dividers on the same parallel, and the other upwards or downwards, as required; read off the latitude at the latter point. Or by parallel ruler (see page 28) lay the edge of the rule on the nearest parallel of latitude, and work it to the given place; note the degree and minute on the graduated meridian at the edge of the rule. For example: In Fig. 7 point C is found to be in the latitude 41° 29′ N. The student should find the latitude of all the points A, B, C, etc.

To Find the Longitude of a Place on the Chart. With the dividers take the distance of the place from the nearest meridian, and refer this distance to one of the graduated parallels, placing one point of the dividers on the same meridian, the other point to the right or left, as required; read off the longitude at the position of the latter point. It may also be done with a parallel ruler in the same manner as the latitude was found. For example: Point C, Fig. 7, has longitude 69° 26′ W (repeat for other points).

To Mark a Ship's Position on a Chart. We will assume that we know the latitude and longitude to be lat. 41° 29′ N, long. 69° 26′ W; with the dividers, take from the graduated meridian the latitude 41° 29′, and mark this up on the meridian nearest to the given longitude (in this case [Fig. 7] the 69° meridian); lay the edge of the parallel ruler on the nearest parallel, and work it to the exact latitude you have marked on the 69° meridian; then with the dividers pick off the given longitude (in this case 69° 26′ W), and lay this distance along the edge of the

parallel ruler which already marks the latitude, and you have the ship's position. In this way a ship's track is usually pricked off at sea, her latitude and longitude being laid down every day at noon, or at any other required time, and the points thus obtained are connected by pencil line, which shows the track of the vessel.

To find the distance between any two places on the chart: Find the latitude midway between the two places on one of the graduated meridians, and mark the point: then with the dividers pick off half the distance between the two places on the chart; now place one point of the dividers on the midlatitude point mentioned, and mark with the other point above and below the midlatitude point, the distance between the upper and lower points being the distance wanted. Now let it be understood that the distance found on the graduated meridian is in degrees and minutes, which may be changed to nautical miles by bearing in mind that one degree of arc equals 60 nautical miles, or one minute of arc equals one nautical mile. The distance in degrees between points B and C is just 0° 60′, which is equal to 60 nautical miles. A clear understanding of the method of using the chart is absolutely essential, for it forms part of the daily routine of a navigator. The method of finding the ship's place on the chart by soundings will be discussed later, in connection with a practical example. For further examples of ways in which to use the chart the student is referred to any of the books already mentioned.

THE SAILINGS.

The principles underlying this subject should be clearly understood, as they will make the remaining portion of the study of navigation much more readily understood, particularly the last of this book.

In considering the position of a vessel at sea, with reference to some other place which she has left or to which she is bound, the following must be considered:

The Course.
Distance.
Difference of Latitude.
Departure.
Difference of Longitude.

Here two cases arise: The first, a vessel goes on a known course and through a known distance from a place whose latitude and longitude are also known, to another place whose latitude and longitude we wish to know. The second, the latitude and longitude of two places being known, it is required to find the course and distance from one to the other.

Plain Sailing is sailing on the supposition that the earth is flat; that the meridians are straight lines parallel to each other and at right angles to the equator. This is of course not true but is sufficiently accurate when very small portions of the earth's surface are considered.

For example, let us suppose a ship to be at A, Fig. 8, and proceeding along A S to S; then A S is the distance in nautical miles (given by the log, see page 27), A N is the departure, N S is the difference of latitude and the angle A S N is the course. Now if any two of these quantities be given, the remaining two can be found by simple trigonometry, or by solving a right triangle. In order to simplify this work Tables I and II in Bowditch's Navigator are used, Table I to be used if the course is in points or fractions of a point, Table II when the course is given in whole degrees.

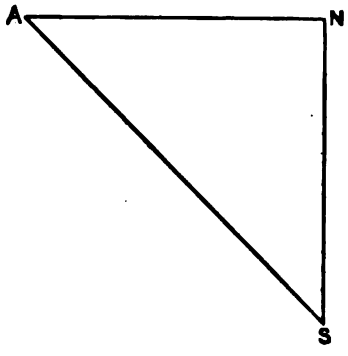


Fig. 8.

Example: A ship is in long. 56° 24′ W and lat. 41° 3′ N at noon on Jan. 1, 1900, and from this position sails 142 nautical miles (given by the log) on a SE by S true course; find the difference of latitude and the departure. In Table I under 3-point courses (SE by S being a 3-point course, as previously explained,) we look up the difference of latitude and departure corresponding to 142 miles distance, and we find diff. lat. = 118.1 miles and the dep. = 78.9 miles. As we have been going in a southeasterly direction, both our lat. and long. have been decreasing. Tables I and II, often called traverse tables, can be used also for solving right triangles.

Parallel Sailing. By this we mean sailing either due east or

west. In this case there is of course no difference of latitude and the distance sailed is the departure. Now if we change this departure from miles to difference of longitude in degrees, minutes and seconds we can at once prick off our position on the chart.

Suppose a ship in lat. 52° N to sail 79 miles due west, the difference of long. is found as follows:

Lat. 52°	Log. sec.	52° = 10.21066
Dep. 79	log.	79 = 1.89763
Log. of Diff. of Long.		= 2.10829

rejecting tens in the index, number corresponding to log 2.10829 = 128.3, which is equal to 2° 8′ 18″, which is the difference of longitude in degrees. Tables I and II may be used for the same purpose, as follows: With latitude as a course and departure in the latitude column, the difference of longitude in minutes is found in the distance column. For instance, in the previous problem take the course as 52°, and use Table II for 52° and we find in the distance column 129 opposite 79.4; in the latitude column $129' = \frac{129'}{60} = 2°\ 9'$ difference of longitude. We might have interpolated in the table, and arrived at the same result as before, namely, 2° 8′ 18″.

Traverse Sailing is merely an extension of the method of plain sailing; and if the plain sailing problem is understood, no trouble will be experienced with traverse sailing.

Mercator's Sailing. First turn to page 15 and read description of Mercator's chart.

The latitude and longitude of two places being known, required to find the course and distance between them: Find the difference between the two latitudes, and reduce it to minutes by multiplying the degrees by 60, etc. Name the difference north or south according to direction in sailing from one place to another. From Table III find the meridional parts due to each latitude and treat them in the same manner as the latitudes, taking their difference or their sum, as the case may be, and call this the meridional difference of latitude. Find the difference of longitude, and name it E or W as the case requires.

Find the logs of the difference of longitude, of the true difference of latitude and of the meridional difference of latitude,

from Table 42, Bowditch. Now from the log of the difference of longitude subtract the log of this meridional difference of latitude, and the result is the log tangent of the course. Look up the angle corresponding to this log tangent in Table 43; this gives the course. Also in Table 43 find the log secant of the course, and add this to the log of the true difference of latitude, and the number corresponding to this sum is the distance. For a more complete account see Bowditch, pages 49–53. The following example may help make the problem clear:

Find by Mercator's sailing the course and distance from Cape Cod lighthouse, in latitude 42° 03′ N, longitude 70° 04′ W, to the Island of St. Mary, one of the Western Islands, in latitude 26° 59′ N and longitude 25° 10′ W.

L_1 =	42°	03′ N.	Merid. Parts	2770.4	Lo_1	70°	4′	W.
L_2	26°	59′ N.	"	2377.6	Lo_2	25°	10′	W.
	5°	4′	m =	392.8		44	54′	

Diff. Lat. = 304′ D. Lo. 2694′

Diff. Lo. = 2694 log. 3.43040
m = 392.8 log. 2.59417

0.83623 = log tangent

Table 42 log. tan. 0.83623 = 81° 42′ = Course

Diff. Lat. = 304′ log. 2.48287
log. sec. 81° 42′ = 0.84056

log. 3.32343 = 2,106 miles dist.

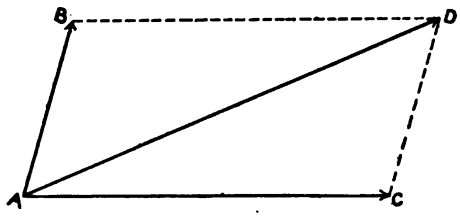
True Course, S 81° 42′ E.
Distance, 2,106 miles.

Formulæ by which all possible cases of Mercator's sailing may be solved are tabulated on page 50, Bowditch.

Great Circle Sailing. The shortest distance between two points on a sphere is along the great circle passing through the points. The simplest way of finding the Great Circle Course is to stretch a string from one point to the other on a globe, note the latitude of the points at which the string cuts the different meridians, and mark upon the chart the positions thus found. Join these points, and you have the course. It is necessary to steer from point to point in order to come nearest to the most direct path between places. On long voyages, such as a trans-

Atlantic or trans-Pacific trip, a great circle track is considerably shorter and is usually taken by steamers. Sailing vessels may find great circle sailing very useful, not only in indicating the shortest distance between two places, but also as showing which is the best tack if the course cannot be laid before the wind. The tack which will keep the vessel nearest to the great circle will be the gaining tack; it may happen that what is apparently a losing tack is in reality a gaining one. Great circle sailing is applicable mainly to steam vessels, for they can generally keep any course desired.

Current or Oblique Sailing. A current is a body of water flowing steadily in a certain direction. The *set* of a current is the direction in which it moves; the *drift* is the rate at which it moves in a specified length of time, as, for example, so many knots per hour. When a ship sails with a current her motion is increased by an amount equal to the drift of the current; when sailing against a current her speed is diminished by an amount equal to the drift. When sailing obliquely to a current a ship's motion is increased if the direction of the current is from abaft the beam, and decreased if from forward of the beam.



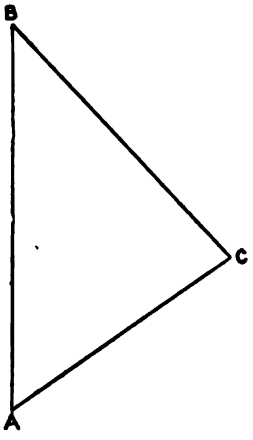
The true course of a vessel sailing in a current is the diagonal of a parallelogram, in which the drift and set of the current in a certain time, and the course steered and speed in the same time, form the adjacent sides. However, in case the set of the current and the course steered are in the same straight line, the current will not affect the course. A simple illustration will show how to find the course if the current is across the path of the ship. Let A be a ship's position, A C the direction of the course steered, and also let A C be the distance in miles the vessel will travel in an hour; further, let A B be the drift and set of a current. Complete the parallelogram A B D C, and the diagonal A D gives the course made good and the distance sailed in one hour.

This is never used when in deep water. The reverse of the

above problem is worked at noon each day when in open ocean, to find the set and drift of the ocean currents. However, in coasting, and where known currents exist, the above may be useful.

Navigating the Ship. *Taking the Departure.* Upon clearing the land, the first step before shaping the course and commencing the log is to fix the ship's position or to take the departure. The position of the ship may be referred to some known point of land, lighthouse, or any prominence whose latitude and longitude are tabulated. The latitudes and longitudes of such places all over the world are tabulated in Table 49, Bowditch.

Departure by a Single Bearing and Distance. Get the compass bearing of the point or lighthouse, and reverse the bearing; this reversed bearing is entered or set down as the first course, subject to the same compass corrections as all compass courses, as previously explained. The distance of the ship from the object, to be estimated by the navigator, is taken as the distance run; this latter is of course somewhat inaccurate, but ordinarily will suffice. The bearing of the object is reversed in order to get the course of the ship, for suppose a light was found to bear by compass NW, then the ship is in a SE direction from the lighthouse. A more accurate method is to take two bearings of the same object; for example, if a ship's track lay so that the object will remain in view for a time, take the bearing of the object and note the angle between it and the direction of the ship's head. After the ship has run on the same course far enough to change the bearing of the object a few points, note again the angle between the bearing of the object and the direction of the ship's head. In the following figure let C be a point of land, and let A be the first position of the ship and B the second; the angle C A B and C B A is obtained as noted above, and the side A B of the triangle is the distance the ship has run between the times of taking the bearings. We may find the necessary distance A C or B C by plane trigonometry, as follows:



$$AC = \frac{AB \sin CBA}{\sin ACB}$$

$$BC = \frac{AB \sin BAC}{\sin BCA}$$

For example, when the ship was at A it was running ENE; after running 25 miles N by W it bore SE: required the distance of the ship from the point of land C at the time of the last bearing at B.

$AB = 25$
$BAC = 7$ points $= 78° 45'$
$CBA = 3$ points $= 33° 45'$
$ACB = 6$ points $= 67° 30'$

$\log 25 = 1.39794$

$\log \sin BAC = \frac{9.99157}{11.38951}$

$\log \sin ACB = \frac{9.96562}{1.42389}$, which corresponds to 26.5

$\therefore BC = 26.5$ miles.

For further examples see Bowditch, pages 61 and 62.

Shaping the course. Having taken the departure, the course must then be shaped, whether by rhumb or by great circle. The true or magnetic course may be taken from the chart; if the magnetic course be taken, allowance must be made for the deviation for that direction of the ship's head to obtain the compass course. If sailing in a known current, proper allowance for it should be made. Should the wind be ahead, the tack must be chosen upon which the greatest distance will be made good toward the point of destination; this latter must be altered, however, should there be any outlying shoals.

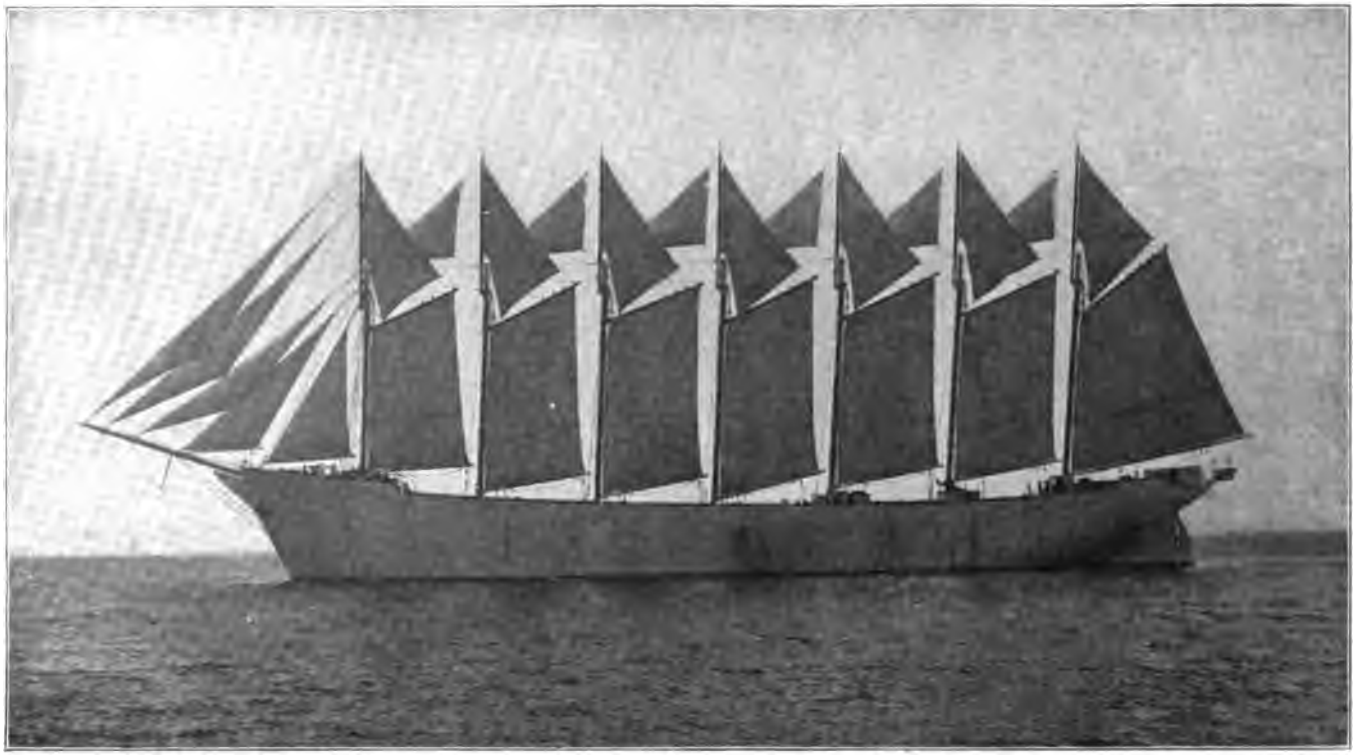
The day's work is the operation of computing from the data expressed in the log-book the ship's run for twenty-four hours preceding each noon. The term is usually restricted to the dead reckoning, and the data given are the latitude and longitude at the preceding noon (by observation), the compass courses, the distance run on each course (given by the log), compass variation, compass deviation for each direction of the ship's head, the leeway, the set and drifts of currents if they are known, the force and direction of the wind, state of sea, sail carried, etc. This data is recorded to find the latitude and longitude by dead reckoning, the true course made good, and the compass bearing and distance

of the port of destination, or that point toward which the ship is to be directed during the coming twenty-four hours.

Strictly speaking, the day's work includes all the computations the navigator must make each day, the results of which are entered in the log-book. In sailing vessels the mate keeps the log-book and the position of the ship by dead reckoning. Dead reckoning is merely the application of the problems already laid down; for instance, if the ship's position is known at noon on a certain day, and during the following twenty-four hours it has run a certain number of miles on a certain compass course, the position at noon on the next day is found as follows: First correct the compass course for variation, deviation, leeway and known currents, to find the true course made good; then in Table I or II, (depending upon whether the course is given in points or degrees) under the distance run, find the latitude made good and the departure. We may change this departure into difference of longitude, as indicated by example in parallel sailing. By applying our difference of latitude and difference of longitude to our position on the preceding day, we get our new position. This method of finding the ship's position is called dead reckoning. On making the land the navigator is aided in his dead reckoning by the lead line, which gives another clue to the ship's position.

Making the Land. Every precaution and forethought should be observed as the ship draws near the land, and the careful navigator will be on the lookout for the many evidences of the proximity of the land, such as the shoaling of the water, the change of color of the water, the change of temperature of the surface water, the presence of land birds, etc. As soon as land is made, the ship's position should be marked off on the chart by reckoning and verified by bearings of well-defined points, if any are in view. An easy and satisfactory method of finding position on making land is to take sextant angles of any well-defined prominent points, which, when used in connection with Lecky's "Off Shore Danger Angles," gives the distance in miles from the point of land to the ship, and by also observing the bearing of the point the exact position of the ship may readily be established.

It is always a good plan to make land, if it can be done without going too far out of one's way, to get a line on the chronome-



The steel schooner Thomas W. Lawson, the largest American sailing vessel. Launched by the Fore River Ship and Engine Company, July 10, 1902.

ters which may or may not have changed their rate. A change in rate of the chronometer or other errors, which will affect the ship's position materially, may thus be ascertained, and should thick weather come on later, the navigator would be in a better position to take the ship safely to her port or destination. Again, should there be any doubt as to longitude, and it was found impracticable to make any point of land, it may be best to make the latitude of the place required and then if possible run on a parallel. The safe way is to locate the ship on a Sumner's line and know to a certainty the relative bearing of the land, which will determine the course; should thick weather be immediately threatening and close in, the navigator would at least know where he is and whether or not danger was near. Of course due regard should always be given to soundings, and even in working your way into an unknown harbor in clear weather, soundings are of infinite value to the navigator. Sumner lines, chronometers, etc., will be taken up shortly and fully explained; also the currents, the time of tide, as well as other facts laid down in the sailing directions, must be considered.

The Method of Sounding. Sounding is the operation of finding the depth of water and the nature of the bottom by means of a lead weight. Two leads are used for sounding,— a hand lead weighing from 7 to 14 pounds, with a line marked to 20 fathoms, and a deep sea lead weighing from 40 to 100 pounds, with a lead line about 100 fathoms in length. A hand lead weighing about 10 pounds is about all one man can easily handle, and is of use only in shallow water; as, for instance, in making a harbor. If the ship is making 7 knots or more, it takes a good leadsman to get bottom with a hand lead in 10 fathoms.

Deep sea leads, as the name implies, are for getting soundings in deep water. It is necessary when using an ordinary deep sea lead of about 40 pounds to heave the ship to; previous to heaving to, the line is reeled off until there is sufficient amount of loose line to reach bottom, and the lead carried forward. The cavity in the bottom of the lead is filled with grease or soap, which catches and retains a sample of the bottom.

Deep sea sounding in this manner is a clumsy operation, and at present is but seldom used. An improved method is the use of

Lord Kelvin's Patent Sounding Machine, shown in Fig. 9. Piano wire wound on a drum is used for the lead line; this makes it possible for the lead to descend much quicker. With this machine it is claimed that bottom can be reached at 100 fathoms, with the ship making 14 knots or more. A glass tube descends with the lead; this tube is hermetically sealed at the top end and open at the bottom. The interior of the tube is covered with a chemical prepara-

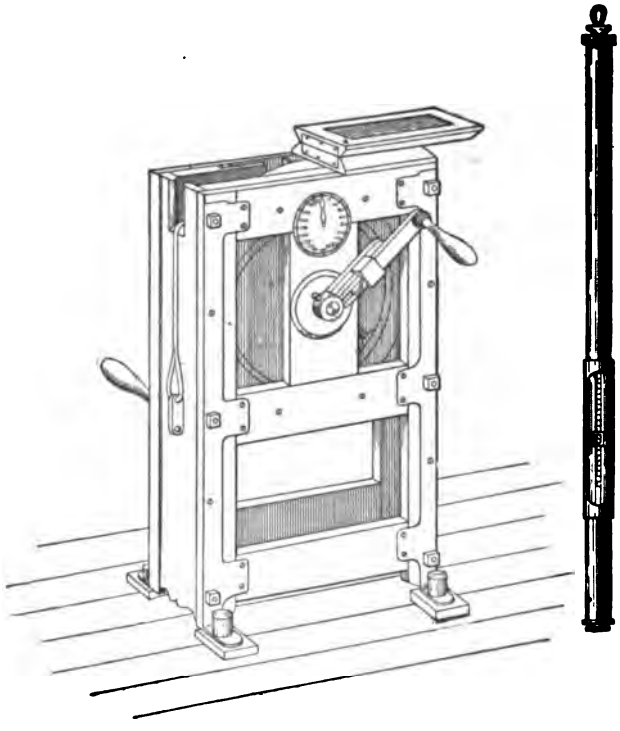


Fig. 9.

tion, which changes color when salt water touches it. As the lead sinks deeper and deeper, the pressure becomes greater and greater, and the air in the tube is compressed as the salt water is forced in. The discoloration of the chemical shows exactly how high the water rose in the tube, and the depth found by means of the scale (graduated to fathoms) attached to the side of the tube. In addition there is a small hand moving over a dial, which also indicates the amount of wire unwound.

For purposes of navigation no importance can be placed upon a single cast of the lead; the proper method to follow in sounding is as follows: Take soundings at regular intervals, say every 4 or 5 miles; mark a strip of paper so that the distance between the marks is 4 or 5 miles, as the case may be, using of course the chart best suited for the purpose, from which the 4 or 5 mile inter-

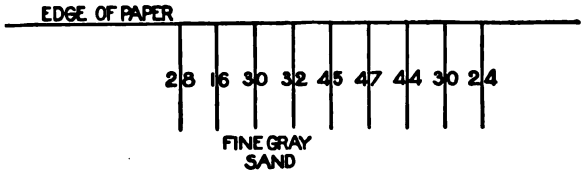


Fig. 10.

vals may be taken. Now as each successive cast is made, the depth and nature of the bottom should be marked on the strip of paper, thus obtaining a series of soundings. The bit of paper will resemble Fig. 10. Now with this strip of paper move about the chart in the locality where the vessel is thought to be, and have the edge of the paper run or point in the same direction as

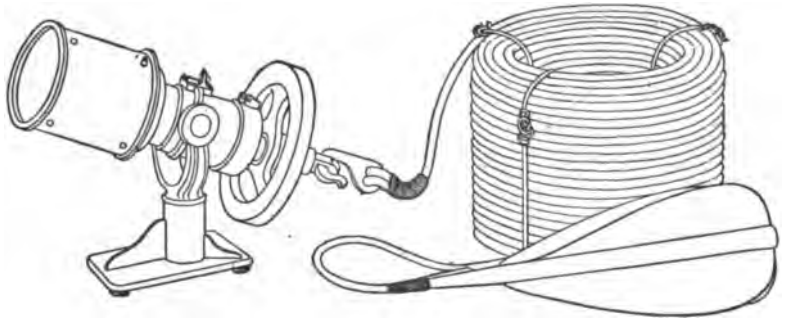


Fig. 11.

the ship is steering true. The soundings on the paper will be found to approximate those on the chart, some places much more closely than others. It requires experience and practice to decide just where the ship is, particularly if it has been running on dead reckoning for several days. Of course in continued thick weather it is more difficult to be absolutely sure of your position by soundings, but nevertheless they are valuable, and may prove that you are not where you think you are.

Taffrail Log. This form of log is now in common use, and shown by Fig. 11. A description will not be given, as it may be found in most books on Navigation and in catalogues; suffice it to say, that the number of knots and tenths of knots are indicated on the dial, and the ship's speed or distance made, ascertained.

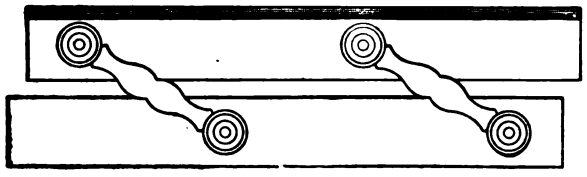


Fig. 12.

The Parallel Rule is shown in Fig. 12; it is used to transfer the direction of lines from one place on the chart to another, as explained in the paragraphs on finding positions, etc., on the chart.

The Horn Protractor, shown in Fig. 13, has to some extent displaced the parallel rule. A form of horn protractor marked in points and degrees, with a long string and hand, as in the figure, is much easier to use than a parallel rule, which at best is troublesome. The object of the string and hand is this: by moving the hand so that the angular distance between it and the string is equal to the correction to be applied to the true course in order to get the compass course, and by observing the index hand we get the compass course to be followed in order to make good the true course. A little practice in the use of one of these patent course finders will reveal their possibilities much easier than they can be described.

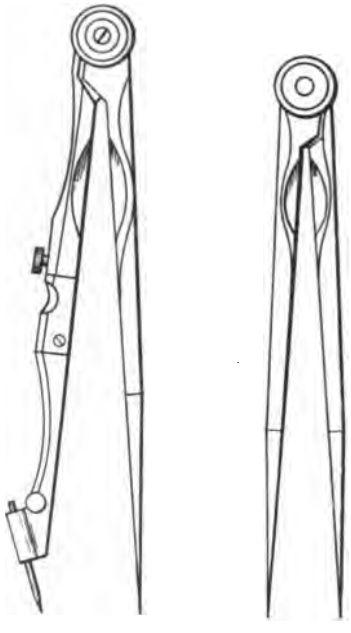


Fig. 14.

Dividers are used for picking off distances upon the chart. It is advisable to have two pairs,— one with both legs metal and one with a pencil point, as shown in Fig. 14.

A word in regard to lead pencils: avoid if possible round

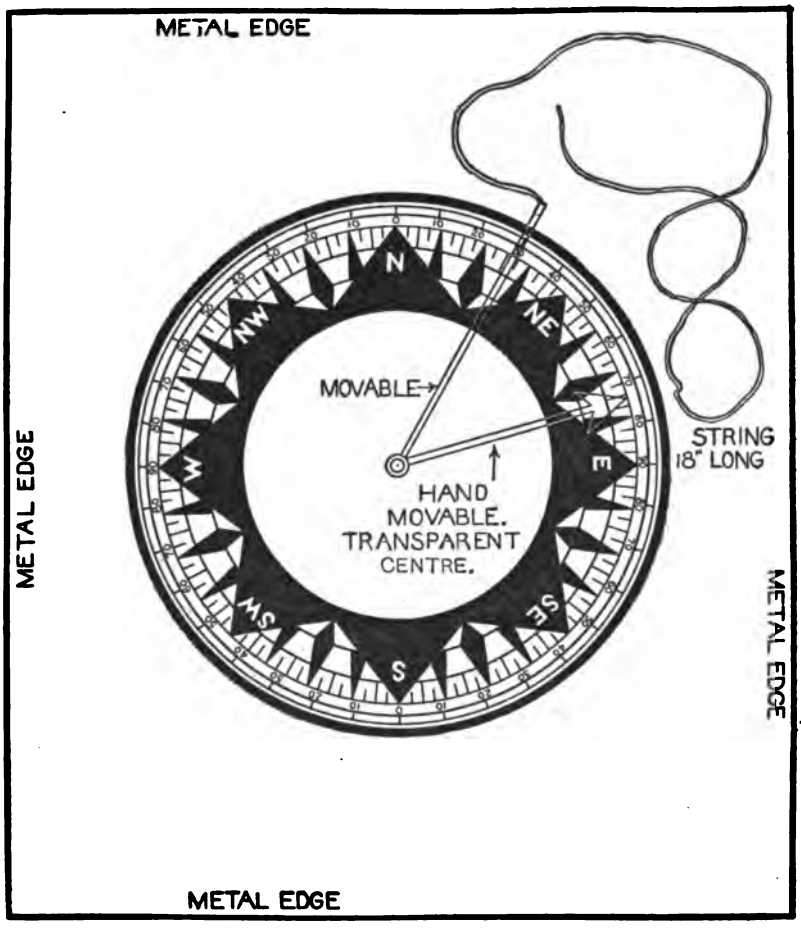


Fig. 13.

pencils; they will be constantly rolling off the chart table at sea, and will thus cause annoyance; either hexagonal or triangular pencils should be used.

The following is taken from Lord Kelvin's "Lecture on Navigation":

"When no landmarks can be seen, and when the water is too deep for soundings, if the sky is cloudy so that neither sun nor stars can be seen, the navigator, however clear the horizon may be, has no other way of knowing where he is than the dead reckoning and no other guide for steering than the compass.

"We often hear stories of marvellous exactness with which the dead reckoning has been verified by the result. A man has steamed or sailed across the Atlantic without having got a glimpse of the sun or stars the whole way, and has made land within five miles of the place aimed at. This may be done once, and may be done again, but must not be trusted to on any one occasion as probably to be done again this time.

"Undue trust in the dead reckoning has produced more disastrous shipwrecks of seaworthy vessels, I believe, than all other causes put together.

"All over the surface of the sea there are currents of unknown strength and direction. Regarding these currents, much most valuable information has been collected by our Board of Trade and Admiralty, and published by the Admiralty in its 'Atlas of Wind and Current Charts.' These charts show, in scarcely any part of the ocean, less than ten miles of surface current per twenty-four hours, and they show as much as forty or fifty miles in many places. Unless these currents are taken into consideration, then the place of the ship, by dead reckoning, may be wrong by from ten to fifty miles per twenty-four hours, and the most accurate information which we yet have regarding them is, at the best, only approximate. There are, in fact, certain currents of ten miles and upwards per day, due to wind (it may be wind in a distant part of the ocean), which the navigator cannot possibly know at the time he is affected by them.

"I believe it would be unsafe to say that, even if the steerage and speed through the water were reckoned with absolute accuracy in the 'account,' the ship's place could in general be reasonably trusted to within fifteen or twenty miles per twenty-four hours of dead reckoning. And, besides, neither the speed through the water, nor the steerage, can be safely reckoned without allowing a considerable margin for error.

"Consider further the steerage. In a wooden ship a good

ordinary compass, with proper precautions to keep iron from its neighborhood, may be safely trusted to within one-half quarter point; but reckoning the errors of even very careful steering by compass, we cannot trust to making a course which will be certainly within a quarter of a point of that desired. Now you know an error of a quarter of a point in your course would put you wrong by one mile to the right or left of your desired course for every twenty miles of distance run. Thus, in the most favorable circumstances you are liable through mere error of steerage by compass, to be ten miles out of your course in a run of only two hundred.

"In an iron ship, if the compass has been thoroughly well attended to as long as the weather permitted sights of the sun or stars, a very careful navigator may be sure of his course by it within a quarter of a point, when cloudy weather comes on; but by the time he has run three or four hundred miles, he can no longer reckon on the same degree of accuracy in his interpretation of its indications, and may be uncertain as to his course to an extent of half a point or more until he again gets an azimuth of sun or star.

"No doubt an exceedingly skilful navigator may entirely, or almost entirely, overcome this last source of uncertainty when he runs over the same course month after month and year after year in the same ship; but it is not overcome by any skill hitherto applied to the compass at sea when a first voyage to a fresh destination, whether in a new ship or an old one, is attempted.

"All things considered, a thoroughly skilled and careful navigator may reckon that, in the most favorable circumstances, he has a fair chance of being within five miles of his estimated place after a two-hundred-mile run on dead reckoning; but with all his skill and with all his care, he may be twenty miles off it; and he will no more think of imperilling his ship and the lives committed to his charge on such an estimate, than a skilled rifle-shot would think of staking a human life on his hitting a bull's-eye at five hundred yards.

"What, then, do practical navigators do in approaching land after a few days' run on dead reckoning? Too many, through bad logic and imperfect scientific intelligence, rather than through

conscious negligence, run on, trusting to their dead reckoning.

"At last the fatal time comes: the navigator has trusted to his dead reckoning once too often; he has made a 'center' not a 'bull's-eye,' and his ship is on the rocks.

"Every seaman of experience will admit the perfect justice of these remarks, even were they not corroborated by the revelations of the wreck register."

NAUTICAL ASTRONOMY.

The preceding portion of this book has dealt mainly with the methods of finding a ship's position at sea by dead reckoning; this method is of course more or less approximate. In spite of the uncertainty, however, dead reckoning must always be kept, as it is sometimes the only available way of locating the ship.

The following portion will be devoted to the branch of navigation which determines the position of a ship by observation of heavenly bodies; this branch is called Nautical Astronomy.

The instruments used in this connection are the Sextant or Quadrant, the Chronometer and the Nautical Almanac. It will now be necessary to give definitions before stating the problems in Nautical Astronomy, previous to reading which the student should reread the definitions given in the first few pages of the book, including those of latitude, longitude, etc.

The following is taken from Norie's "Navigation."

"The Solar System is that assemblage of planets, comets, meteors and cosmical bodies that have the sun as their center luminary, revolving around that body at different distances and with different periodic times.

"The planets of the system, in the order of their proximity to the sun, are: Mercury, Venus, Earth, Mars, the Asteroids, Jupiter, Saturn, Uranus and Neptune. The planets shine by the reflected light of the sun, as do all the bodies of the system; and it is characteristic of planetary motion that a line drawn from the sun's center to a planet would sweep over equal areas in equal times. The inferior planets are the two whose paths are between the earth and the sun; the superior planets move in paths at a greater distance from the sun than the earth's path. A satellite is another name for a moon, and most of the planets in the solar

system are accompanied by one or more such bodies, which revolve around their primary, as a secondary planet. The earth has one moon; Mars two; Jupiter five (the fifth is seldom seen); Saturn eight, and also meteoric rings; Uranus four, and Neptune one or more.

"Comets are luminous bodies whose very eccentric paths lie around the sun.

"Fixed stars are distinguished from the planets inasmuch as their relative positions in the heavens remain the same from year to year. Unlike the planets, which shine by reflected light of the sun, the fixed stars shine by their own light the same as our sun, and are supposed to be the centers of other solar systems. The only fixed stars used in navigation are those of the first and second magnitude. The distances of a few of the fixed stars have been determined; of these distances, none are less than 200,000 times the distance of the earth from the sun. In other words, light travels at the rate of 186,000 miles a second, and it takes eight minutes for the light to come to the earth from the sun; to come from the nearest fixed star α (centauri, a southern star) it would take three years.

The celestial bodies used for navigation purposes and from which data are derived from the Nautical Almanac and Sextant, are the Sun, Moon, Mars, Venus, Jupiter, Saturn and the fixed stars of first and second magnitude. In this book the sun only will be considered, but the student should understand that the stars are used in practically the same manner as the sun, and if he can use the sun he can also use the stars. There is a feeling current among seamen that there is something mysterious about stars, and that it requires a college education to successfully study them; this is not so. For stellar work in navigation, if one is to study by himself, Lecky's "Wrinkles in Practical Navigation" is doubtless the best book.

Nautical Astronomy. To a person on the surface of the earth the heavens appear like a huge hollow sphere the center of which is the center of the earth. This idea of a large, hollow sphere is more apparent when one is at sea. Only one-half of this sphere is visible to a person on the earth's surface. If a plumb line were drawn through the person and extended far enough in both direc-

tions, it would pass through the center of the earth and in the other direction to the surface of the supposed hollow sphere.

Now a surface or plane perpendicular to the plumb line and extending on all sides to meet the concave celestial surface, would divide the hollow sphere into two parts; this plane is called the Rational Horizon. In Fig. 15 let circle E represent the earth and the large circle H Z′ H′, etc., the above-mentioned hollow sphere, or celestial concave, as it is called; let P be the north and T the south pole; T P is then the axis of the earth. P′ or T P prolonged until it cuts the celestial concave is the elevated pole, so called. Let Z be the position of an observer standing upon the surface of the earth; then, as already defined, H H′ is the rational horizon, the circle Q′ Q′ at right angles to the axis T P P′ is the equator on the earth, the celestial equator being Q Q. Hereafter all circles mentioned will be on the celestial concave and not on the earth, although the corresponding circles on the earth, if one cares to consider them, are similar in every respect. For instance, the latitude of a point in the celestial concave is the same as the latitude of a point similarly situated upon the earth. Take point Z on the earth (Fig. 15), and Z′ the similar point in the celestial concave, or zenith point; they both have the same latitude and longitude, because their location is given in degrees, etc., and there are just as many degrees in the arc Q Z′ as there are in arc Q′ Z. As Q Q is the equator, the latitude of Z is the arc Q′ Z or Q Z′ expressed in degrees, minutes and seconds, as defined on page 5. The circle H Z′ H′ N is the observer's meridian. Let S be the position of the sun in the heavens at any given time; let P′ S R M be another meridian passing through the sun's position and making the angle *o* with the observer's meridian; furthermore, let circle Z′ S A N be a great circle, also passing through the sun's position

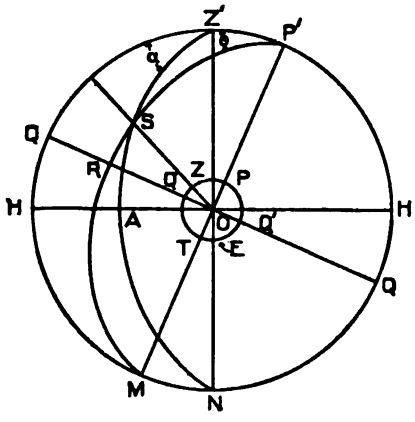


Fig. 15.

S and the zenith Z′, making the angle *a* with the observer's meridian. Angle *o* is called the *hour angle* and the angle *a* the sun's azimuth. Angle *o* is generally expressed in hours, minutes and seconds, and is the apparent time of the observer at Z on the earth (see explanation of time, page 44). Reference to Fig. 16 in connection with Fig. 15 may make these definitions clearer.

Fig. 16 is a view of the celestial concave looking from Z, the observer's position on earth, up to the zenith directly overhead; the same letters are used and refer in both figures (Figs. 15 and 16) to the same points. R S is the sun's declination, or its distance north or south of the equator in degrees, etc. A S is the true altitude of the sun, or its distance above the rational horizon; Q Z′ is the latitude of Z or Z′; the longitude has already been defined. S P′ is the sun's polar distance, S Z′ the sun's zenith distance and Z′ P′ the colatitude of the observer's position at Z. Z′ N in Fig. 15 and Z′ N′, Fig. 16, is the prime vertical or vertical circle passing through the east or west points.

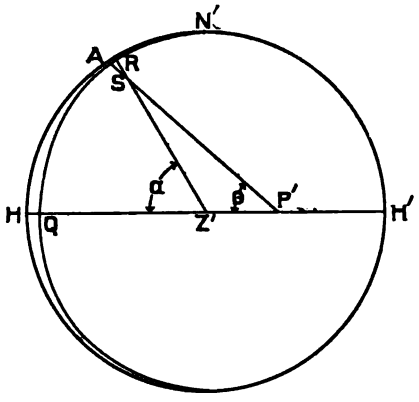


Fig. 16

The student should learn and know what the following terms mean; their use will appear later. They are all essential in the work of navigation, declination, altitude, zenith distance, polar distance, celestial meridian, azimuth, hour angle, colatitude, etc.

Declination is the distance of the sun or any celestial body from the equator, given in the Nautical Almanac (see page 50), for every day in the year.

Altitude is the distance of any celestial body above the horizon, and is measured by a sextant.

Zenith distance is 90° minus the altitude.

Polar distance is 90° minus the declination.

Hour Angle. At three o'clock, for example (or to be more

correct, three hours apparent time), the sun's hour angle is three hours; that is, it takes three hours from the time the sun crosses the meridian (which it does at noon) until it reaches the position where its hour angle is three hours.

Colatitude is 90° minus the latitude.

Azimuth of a body is the angle at the zenith between the observer's meridian and the circle of altitude as per Fig. 15; when on the prime vertical the azimuth of a body is 90°.

Amplitude is the arc of the horizon between the point where

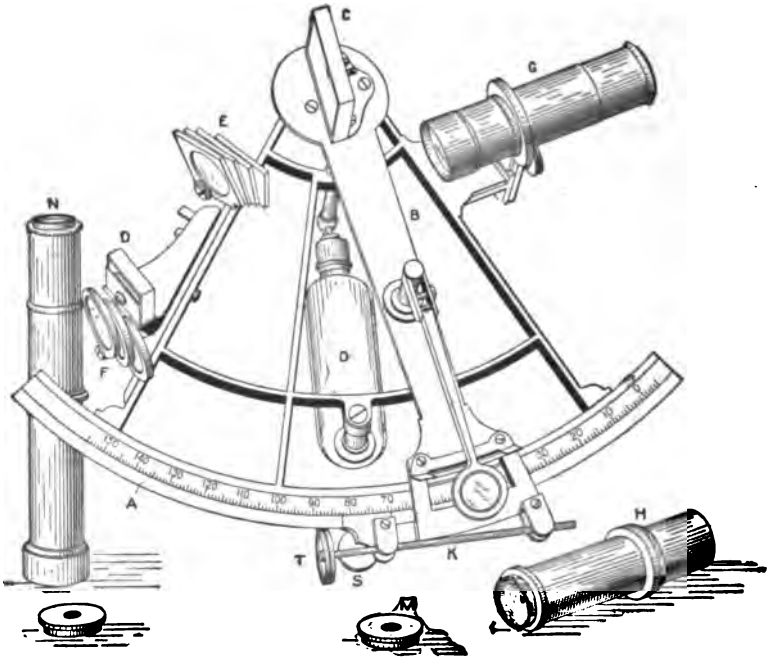
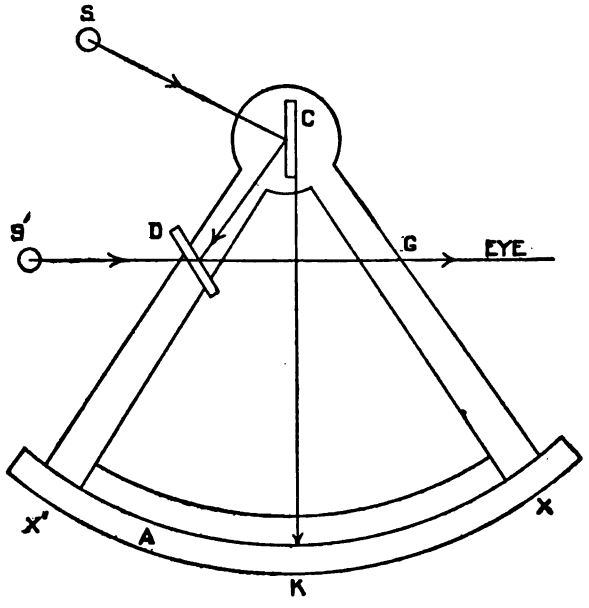


Fig. 17.

the sun rises or sets and the east or west point; it is the complement of the azimuth.

The Sextant. This instrument, shown by Fig. 17, is used for measuring angles between two points by bringing the reflected image of one of the points to coincide with the other as seen directly. The frame of the sextant is usually made of brass or gun metal. C is the index glass, D the horizon glass, half of

which is silvered. A is the arc, and is divided into degrees, which are generally subdivided to ten minutes; the movable arm B carries at K a vernier by which still finer readings may be had; the verniers are generally divided so that the angle may be read to twenty seconds. G is a telescope which screws into a ring which holds it; N is an inverting telescope which is often used in place of G; H is an eyepiece without lenses, and also screws into the telescope ring; M is a colored eyepiece which goes with the telescope; E and F are colored screens, used when the sun is bright; O is a wooden handle by which the instrument is held.



The principle on which the sextant works is this: If a ray of light suffers two successive reflections in the same plane by two plane mirrors, the angle between the first and last direction of the ray is twice the angle of the mirrors. D is set parallel to C X, and mirror C turns with arm B. The angle between S C and S′ G is twice the angle between the two mirrors D and C; then if the angle X′ C X is 60°, the divisions on the scale A in the same length of arc X X′ will be 120°. The use of the sextant is as follows: We wish to find the angle between two points S and S′.

Hold the instrument by the handle and look directly at S′, which is seen through the unsilvered portion of the horizon glass; now move the arm B until point S is seen in the silvered portion of the horizon glass, and the angle between the two objects is indicated at K. This statement is a crude one, and is simply intended to show what the instrument does rather than giving the exact methods of using, which are carefully stated in standard works on Navigation. A few points regarding the instrument are well to know, as every intelligent navigator should be able to "size up" his sextant.

The vernier: When the zero line is in coincidence with a line on the scale, the terminal line should also be in coincidence with another line on the scale; furthermore, in every well-cut arc only one line on the vernier appears to coincide with a line on the scale.

It is not advisable to lay a sextant down in the hot sun; the metal of the arc being different from that of the sextant frame, unequal expansion takes place and may result in a permanent distortion of the divided arc, thus destroying the accuracy of the sextant.

The horizon glass should be perpendicular to the plane of the sextant, as should the index glass, also. The line of sight of the telescope should be parallel to the plane of the instrument. If there were no index error, the zero or first graduation on the scale should coincide with the first mark on the vernier. Index error is common, and can be found roughly by making the image of the sea horizon coincide with the horizon itself as accurately as possible. If the zero of the index is at zero on the scale there is no index error, but if it is on the arc the correction is so much subtractive, or when off the arc is so much additive. A better way to do it is to measure the sun's horizontal diameter; as this method is explained in any good book on navigation, it will not be given here. The common use of the sextant is to get altitudes of heavenly bodies, to measure heights and to measure angles or the distances between points in general.

When a sea horizon cannot be obtained, that is, when a ship is in port, the altitude of a body can be found by using an *artificial horizon* (which consists of a saucer of quicksilver or molasses)

instead of depending upon a sea horizon. The process consists in bringing the image of the sun down to the image reflected from the artificial horizon. The altitude observed in this way is twice the apparent altitude.

Perhaps the best method for a novice to pursue in observing an altitude, is to start with arm B, at zero and look directly at the sun; both the real sun and its reflected image will appear in the horizon glass D; now slowly bring the instrument down until the sea horizon is seen in the horizon glass, moving arm B at the same time, keeping the sun's reflected image in the horizon glass; when the lower limb of the sun is about to touch the horizon, clamp the arm B by the screw S, and to get the final contact between the sun's limb and the horizon turn the tangent screw T. Practice makes perfect; this is true of taking altitudes or in using the sextant generally. Personal instruction by some one familiar with the use of the instrument will enable one to get reliable observations sooner than by stumbling along alone.

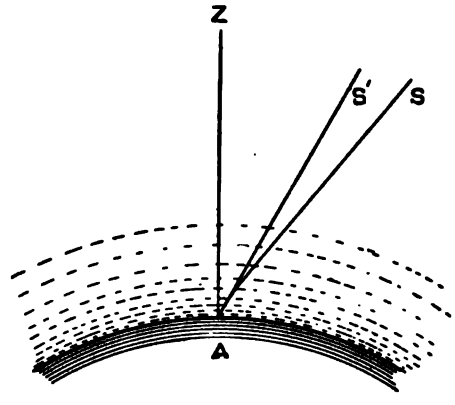


Fig. 18.

True Altitude. The altitude of a celestial body as read off a sextant is not the true altitude, but is subject to corrections. First comes the index *error* of the sextant, which has been explained; the next is *dip*, which allows for the height of the eye above the surface. Table 14, Bowditch, gives the dip correction for varying heights of eye; dip is always a subtractive correction.

Refraction. Owing to the atmosphere which surrounds the earth, the sun's rays are bent, as indicated by Fig. 18; hence, to a person standing at A and looking at the sun, which is really at S, it appears to be at S′. Refraction is a subtractive correction, and is found in Table 20, Bowditch, it depends upon the altitude of the body and the density of the atmosphere (height of the barometer).

Parallax is due to the fact that we observe the sun from the earth's surface and not from its center. To an observer on the earth the sun appears in the direction B S, Fig. 19, but its direction from the center of the earth is O S, hence as we are reducing the observed altitude to the true altitude we must correct for

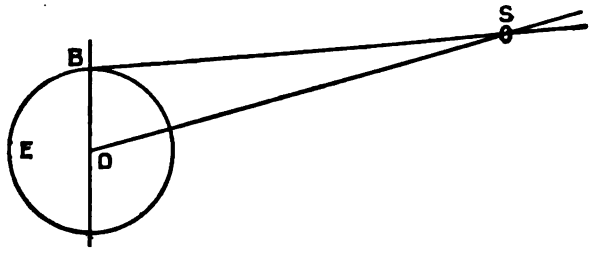


Fig. 19.

parallax an additive correction, which is the angle B S O, and is given in Table 16, Bowditch.

Semidiameter of a celestial body is half the angle subtended by the body's diameter. This is an additive correction if the lower limb is used, and gives the altitude of the body's center; this correction is given in the Nautical Almanac.

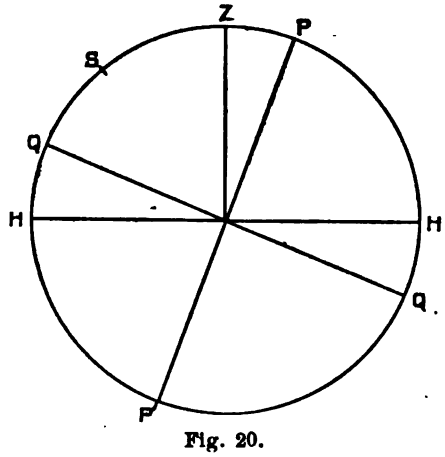
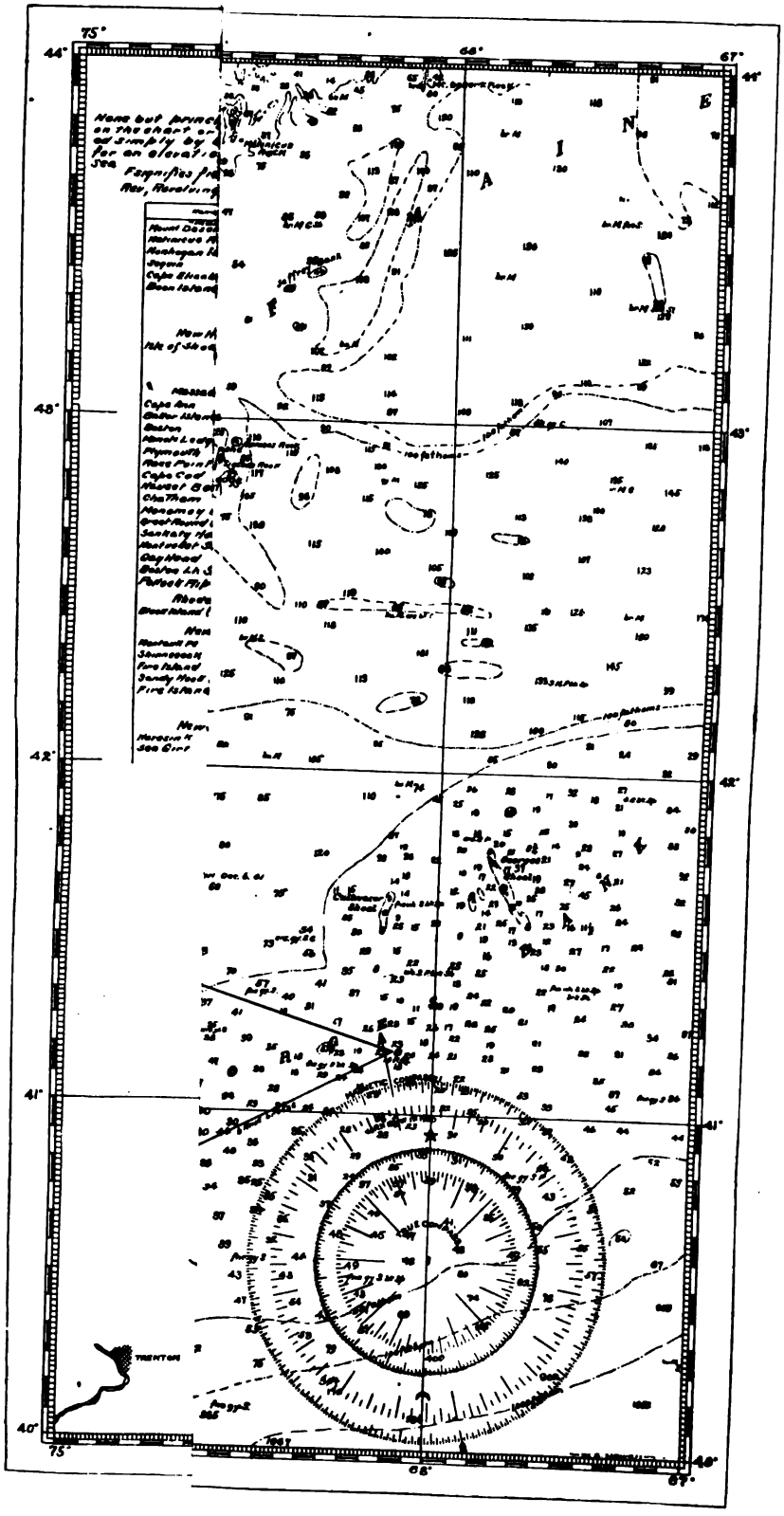


Fig. 20.

Finding Latitude. When the body observed is on the meridian its altitude ceases to change, and the most convenient way of finding the latitude is by observing a meridian altitude. Referring to Fig. 20 let H Z H, etc., be the celestial concave, Z the zenith, P the elevated pole, S the position of the sun or other celestial body, Q Q the celestial equator, and H H the rational horizon.

First let us see what we know: we can get the declination Q S of the sun or other bodies, from the Nautical Almanac; we can also get H S, the true altitude, by correcting an observed


75°
44°
68°
67°
45°
43°
42°
41°
40°
TRENTON

altitude made with a sextant. The latitude we are after is the arc Q Z; we know that A Z is 90 degrees, as the zenith is 90 degrees from the horizon, therefore S Z is equal to 90 degrees — S H = 90 degrees — true altitude, and is called zenith distance. Furthermore, Q Z = latitude = Q S + S Z = declination + zenith distance. A concrete case will illustrate this. The student is also referred to a sample page from the Nautical Almanac at the back of the book.

At sea, May 3, 1902, long. 45 degrees W. obs. meridian, alt. sun's lower limb (⊙) 56 degrees 10′, sun bearing south, index corr. + 2′ height of eye 20 feet: required the latitude.

44.5	Decl.		15° 28′ N
8	Corr. 45° W		+2′
133.5″ = 2′ +	Red. Decl.		15° 30′ N
Index Corr. + 2′			
Semidiam. + 15′ 53″	obs. alt.	56° 10′	
dip —4′	corr.	+ 13′	
refraction —1′	true alt.	56° 23′	
	zenith dist.	33° 37′	33° 37′ N
parallax + 5″	latitude		49° 7′
+ 18′ — 5′			

Take the sun's declination from the Nautical Almanac for noon of the day and correct it for the longitude, because the given declination is for the Greenwich mean noon. The hourly difference is also found in the Nautical Almanac; remember that 15 degrees of longitude is the same as one hour; add the hourly difference if the longitude is west, subtract if the longitude is east. Correct the observed altitude for index error, dip, refraction, semidiameter and parallax, and subtract it from 90 degrees to get zenith distance. When the observer is north of the sun call the zenith distance N, when he is south of the sun call it S. When the declination and zenith distance are of the same name take their sum; when of opposite names, take their difference — the result is the latitude. When the declination and zenith distance are of the same name, the latitude is also of that name, when of different names the latitude takes the name of the greater. In low latitudes the sum of corrections for dip, parallax, semidiameter and refraction is practically + .12′; therefore, if we subtract the

sextant altitude from 89 degrees 48′ or 90 degrees minus 12′, we get the same result as if we added 12′ to the correct sextant reading and then subtracted from 90 degrees. This practice of subtracting the correct sextant reading from 89 degrees 48′ to get zenith distance is common in sailing vessels; it should, however, not be used in high latitudes or low latitudes; in fact, until one understands the nature of the corrections, it is best not to use 89 degrees 48′, but to apply each correction separately. If the correction for parallax is left out (in the case of the sun or fixed stars) no serious error will be introduced.

Latitude by Reduction to the Meridian. The following concise explanation is taken from Raper.

"When the sky is cloudy, or the weather variable, the sun or any other celestial body, though obscured when exactly on the meridian, frequently appears for short intervals of time, both before and after the meridian passage. When the body is near the meridian, the change of altitude in a small portion of time is very small, and though the altitude near the meridian changes at a different rate in different latitudes, yet the change of altitude in a given small interval is not sensibly affected by a change of several miles in the latitude, and therefore it may be computed with tolerable accuracy, even when the latitude by account (which is used in the computation) is considerably in error. If, accordingly, at the time of observing an altitude near the meridian, we know the hour angle, we may find very nearly by computation the difference of altitude by which to reduce the observed altitude to the meridian altitude, and which is thence called the Reduction to the Meridian."

"This method is, in point of simplicity, but little inferior to the meridian altitude, to which it is next in importance; and it particularly demands the attention of seamen, because, when the latitude by observation is left, as it too generally is, to the casualty of obtaining the meridian altitude, it is frequently lost for the day."

"The term 'near the meridian' implies a meridian distance limited according to the latitude, the declination, and also to the degree of precision with which the time is known."

The rule for these limits is as follows: The hour angle of the

sun, or the time from noon, should not exceed the number of degrees in the sun's meridian zenith distance. For instance, if the sun's meridian altitude be about 75 degrees, the time from noon of taking the ex-meridian should not be greater than fifteen minutes, but better less.

The computation. Take an altitude of the sun and note the time. Find the sun's hour angle, or the time from noon; thus, when the apparent time has been lately determined, that is, by the time sight in the morning (see discussion on time, page 46). If the ship has made westing, subtract the difference of longitude made good from the apparent time found; if easting has been made, add the difference of longitude to the apparent time, the result is the apparent time required.

If the apparent time has not lately been found by observation, the apparent time may be found from the chronometer, using the longitude by account, thus: To the G. M. T. (Greenwich mean time) given by the chronometer (after applying the gain or loss up to the time) reduce the equation of time (Nautical Almanac) and apply it to the G. M. T. as indicated in the Nautical Almanac. The result is the Greenwich apparent time. If the longitude is west, subtract this longitude in time from the G. A. T. (Greenwich apparent time), or if east add it, and the result is the apparent time at ship.

Find the declination of the sun, and correct the altitude for dip, semidiameter, parallax and refraction and index error. Add together the logarithm from the table, Bowditch, and the log, sine square of the hour angle; the sum is the log sine of the reduction. Add the reduction to the true altitude, and the result is the meridian altitude at the place where the meridian altitude was observed; use this meridian altitude to find the latitude of the ship as previously indicated. The latitude thus found is the latitude of the place where the observation was made, and not at noon; to get it from noon we must use Tables I and II, Bowditch, taking into account the course made good and the distance run during the time between the observation and noon at ship.

Example: Latitude by account 47 degrees N, longitude by account 25 degrees W at 11 minutes 48 seconds before noon,

corrected altitude of sun 63 degrees 54′, reduced declination 17 degrees 4′N, find the latitude.

Lat. 47° decl. 17° (both N) 0.416
11m 48s sine sq. 6.821
log. sine of reduction = 7.237 red. = 0° 6′
alt. 63° 54′
mer. alt. 64° 0′

90° 0′
64° 0′ merid. alt.
26° 0′ N zenith dist.
17° 4′ N Declination
43° 4′ N Latitude 11m 48s before noon at ship.

The Chronometer. This is a watch constructed as nearly perfect as possible; it is furnished with an apparatus by which the changes of rate due to expansion or contraction, by heat or cold, are nearly obviated. Chronometers should receive great care and be placed in a position where no jars or shocks are likely to be experienced, as these will often change the rate. This instrument gives Greenwich mean time (when the correction is applied) and is, of course, a factor in the determination of longitude.

A chronometer which has a regular rate, that is, gains or loses the same amount every twenty-four hours, is satisfactory, but when this rate varies and no dependence can be put upon it, the instrument is of little or no value. Chronometers often change their rates, that is, after going for some time gaining or losing a certain amount each day, they gain or lose another amount for a while. The navigator should watch his chronometers carefully and compare them (if two or more are on board) every day; further, should any points of land or islands come in sight, so that the ship can be located by bearings of the shore, the chronometer error can be ascertained (this will be taken up later).

Relation of Longitude (Arc) to Time. The following will show just how the chronometer is the agent by which we find our longitude. The circumference of every circle is divided into 360 equal parts or degrees; now the earth, whose circumference is a circle, is 360 degrees, turns on its axis in the direction of this circumference once in 24 hours; therefore the 360 degrees of circular measure are the equivalent of 24 hours; dividing 360 degrees by 24 we get 15, hence 15 degrees of arc are the equivalent of one

hour. We say, astronomically, that as a complete rotation of the earth upon its axis is performed in 24 hours, meridians 15 degrees apart are thus brought to the sun at regular intervals of one hour.

In circular measure	(arc)	In Time
360 degrees	=	24 hours
15 "	=	1 hour
1 "	=	4 min.
15′	=	1 min.
1′	=	4 sec.
15″	=	1 sec.

By means of this table we can change longitude into time, or time into longitude.

Time. The instant that a body is on the meridian of the observer, it is called its transit or culmination. A body is, in the course of one complete turn of the earth, on the meridian twice; when on the meridian directly over the observer's head it is called the upper transit; when underneath, the lower transit.

Solar Time is measured by the daily motion of the sun, and is called apparent time. Apparent time is the hour angle of the center of the actual sun.

The Apparent Solar Day is the time between two successive upper transits of the sun. When the sun is on the observer's meridian, it is *apparent noon* at that place.

The apparent day varies in length from two causes: *first*, the sun does not move in the equator but in the ecliptic, the plane of the ecliptic making an angle with the plane of the equator equal to the inclination of the earth's axis; *second*, the sun's motion in the ecliptic is not uniform. At times the sun moves over an arc of 57′ of the ecliptic in a day, and at other times over an arc of 61′ in the same time.

Thus apparent time cannot be taken as a standard, and for most purposes time is measured by a fictitious sun or "mean sun" which moves in a uniform rate along the equator. This is called *mean solar time.* The difference between mean and apparent time is called the *equation of time.*

To convert apparent time into mean time, take from the Nautical Almanac the equation of time and add it or subtract it according to the direction given at the column. The equation of time is sometimes defined as a quantity to be added algebraically

to apparent time to get mean time. Its sign is plus when mean time is later than apparent time, and minus when earlier.

Siderial Time is measured by the daily motion of the stars, or the daily motion of the vernal equinox.

Siderial Day is the interval between two successive upper transits of the vernal equinox over the same meridian.

Siderial Noon is the instant the circle through the vernal equinox coincides with the meridian.

In order to connect siderial time with mean time an element is calculated and tabulated in the Nautical Almanac, that is, the Siderial Time of Mean Noon, or what is the same, the Right Ascension of the Mean Sun. Hence the Siderial Time as found by the sun is,

Siderial Time = Right Ascension Sun + Hour Angle of Sun.

When the hour angle is 0 the Right Ascension is the Siderial Time.

Astronomical and Civil Time. In astronomical time the day begins at noon and is divided into 24 hours, numbered successively from 0 h. to 24 h. The civil day begins at midnight, or when the sun is at the lower transit. It is divided into two parts of 12 hours each, the part from midnight to noon being called A.M. and from noon to midnight P.M. The civil day begins 12 hours before the astronomical day of the same date.

Ast. Time August 20, 17 hours = Civ. Time Aug. 21, 5 hours A.M.

Ast. Time August 20, 7 hours = Civ. Time Aug. 20, 7 hours P.M.

The hour angle of the sun at any meridian is called the local (Solar) time. The hour angle at Greenwich at the same instant is the corresponding Greenwich time. The difference between these two times is equal to the longitude of the place in time, which may be changed to degrees, minutes and seconds by the table previously given. Thus the longitude (generally speaking) is found at sea by getting the sun's hour angle, by aid of a sextant, and at the same time noting the chronometer time, the difference between the two being the longitude of the place of observation in time.

The Nautical Almanac, a portion of which is inserted on

pages 50 and 51, gives part of the data needed to work up a sight. The Almanac is arranged as represented here, with the right ascension, declination, etc., of the sun corresponding to Greenwich apparent noon on this left-hand page and those corresponding to Greenwich mean noon on the right. The table for mean noon is the one to use, as we get Greenwich mean time from the chronometer.

Longitude at Sea by the Sun. In order to solve this problem it is necessary to know the following: Greenwich mean time at the time of observation, the sun's true altitude, declination, and the latitude of the place of observation. It is customary at sea to take a time sight in the morning about nine o'clock, and to work it through, using for latitude the latitude by account (which is obtained by the aid of the log and traverse table, as previously explained), thus getting the apparent time at ship and also noting the time by the ship's clock (not by the chronometer); thus the ship time for the day is found. Should the sun be in the prime vertical at the time of taking its altitude, the latitude by account may be used then and there to work out the ship's longitude, for with the sun in the prime vertical an error in the latitude produces no error in the longitude; otherwise the longitude must not be reckoned until the latitude of the ship is found at noon. Having obtained the latitude by meridian altitude, the latitude at the time of the morning observation may be found and used in working up the longitude. It is a good plan to get the sun in the prime vertical if possible; however, the altitude should not be too small to be seriously affected by refraction.

Let us now carry through a complete case, with every step plainly indicated, which the student must follow carefully.

The usual method followed is given by the following rule: Add together the true altitude, the latitude, and the polar distance; divide this sum by 2 and subtract the altitude; find from table 44, Bowditch, the log sec. of the latitude; the log cosec. of the polar distance; the log cos. of $\frac{1}{2}$ the sum of the altitude, latitude, and polar distance, and the log sin. of $\frac{1}{2}$ the sum minus the altitude; add these from logs and divide the sum by 2, and the quotient is the log sin. of $\frac{1}{2}$ the hour angle. The steps are indicated in the following practical case:

Example: At sea, May 6, 1902, the ship's position being lat. 41° 25′ N, long. 7° 320′ W by account (dead reckoning) from position at noon on May 5, the following observations were made:

Sun's altitude, lower limb		29° 4′
Time by ship's clock		$7^h\ 55^m$
Chronometer		$0^h\ 7^m\ 51^s$
Log		15
Course W N W true		
Chronometer correction		$+\ 4^m\ 59^s$
Index error of Sextant		$-\ 1'$
Barometer		30.32 inches.
Thermometer		59°
Height of eye		17 ft.
Chronometer, May 6	$0^h\ 7^m\ 51^s$	
Chronometer correction	$+\ 4^m\ 59^s$	
	$0^h\ 12^m\ 50^s$ = Greenwich.	

Mean time May 6 (astronomical time).

Alt. (sun's) ⊙	29° 41′ 0″	Index error		− 1′ 0″
Correction (A)	9′ 16″	Dip, Bowditch, Table 14		− 4′ 2″
True alt. sun's center	29° 50′ 16″	Refraction, Bowditch, Table 20		− 1′ 42°
		Semidiam. Nautical Almanac		+ 15′ 52′
		Parallax, Bowditch, Table 16		+ 8″
		Correction (A)		+ 9′ 16″
Declination May 6, Nautical Almanac.		16° 20′ 48.5 N	Hourly diff.	+ 42.57″
		8.5	corr. for 12m.	+ 8.5″
Corrected Decl.		16° 20′ 57″ 0		
Polar Distance		73° 39′ 3″		
Equation of Time Nautical Almanac		$3^m\ 26.23^s$ added to mean time.		[The correction here for 12m. is too small to take account of.]

At noon, May 6, the latitude by meridian altitude was found to be 41° 40′ N, the log read 33 and the course steered during the morning was W N W, true. The distance run from the time of morning sight at 7:55 to noon was given by the log 33 — 15 = 18 miles. Table I, Bowditch, for W N W course, distance 18, gives diff. lat. 6.9; departure 16.6; diff. long. 22′.

∴ Lat. by account at 7:55 A.M. is 41° 4′ + 6.9′ = 41° 33.1′ or 41° 33′.

This latitude is to be used in working out the morning sight.

Alt.	29° 50′ 16″		
Lat.	41° 33′ 0″	log sec.	0.12588
Polar Dist.	73° 39′ 3″	log cosec.	0.01793
Sum.	145° 2′ 19″		
½ Sum.	72° 31′ 9″	log cos.	9.47767
	29° 50′ 16″		
½ Sum. − Alt. =	42° 40′ 53″	lon. sin.	9.83120
			2)19.45268
$7^h\ 42^m\ 35^s$ A.M., log ½ sin t			9.72634

The ship's clock gave the time as 7.55 when the time by the sun was found to be 7.42, which makes the ship's clock 13 minutes fast.

This is the apparent time at ship and is found in Bowditch, Table 44, corresponding log sin. 9.72634. As the chronometer (corrected) gives mean time it is necessary to find the mean time at ship at the time of observation, as follows:

$7^h\ 42^m\ 35^s$ apparent time
$-3\ \ 26$ Eq. of time
$7^h\ 39^m\ \ 9^s$ mean time at ship.
12
$19^h\ 39^m\ \ 9^s$ astronomical mean time at ship, May 5.

Greenwich mean time (astronomical) $0^h\ 12^m\ 50^h$, May 6.
$0^h\ 12^m\ 50^s$ May 6 = $24^h\ 12^m\ 50^s$ May 5.
$19^h\ 39^m\ \ 9^s$ May 5.
Subtract $4^h\ 33^m\ 41^s$ = long. in time.
68° 25′ 15″ W long. at 7.55.
22 diff. of long.
68° 47′ 15″ W longitude reduced to noon, May 6.

In practice it is customary to work up the time sight, using the latitude by account, thus finding at once the approximate correction for the ship's time. It must, however, be worked through again to get the longitude as indicated by the problem. It requires practice to enable one to handle the longitude problem expeditiously.

A short method of checking the longitude sight is to get an altitude of the sun before its meridian passage, say at eight or ten minutes before; note the chronometer time at the instant of observing the altitude. Watch the sun, and when it again comes to the same altitude note the chronometer time; the mean of the

chronometer times gives the Greenwich mean time of apparent noon; this Greenwich mean time must be corrected for the equation of time, and when thus corrected it gives the longitude in time, which is of course changed into degrees, minutes and seconds.

Page from the Nautical Almanac giving right ascension, declination, etc., of the sun corresponding to Greenwich apparent noon.

AT GREENWICH APPARENT NOON.

Day of the Week.	Day of the Month.	THE SUN'S					Sidereal Time of Semi-diameter Passing Meridian.	Equation of Time to be Subtracted from Apparent Time.	Diff. for 1 Hour.
		Apparent Right Ascension.	Diff. for 1 Hour.	Apparent Declination.	Diff. for 1 Hour.	Semi-diameter.			
		h m s	s	° ′ ″	″	′ ″	s	m s	s
Thur.	1	2 31 00.58	+ 9.528	N. 14 52 24.0	+ 45.78	15 53.28	65.92	2 53.92	0.328
Frid.	2	2 34 49.53	9.551	15 10 35.4	45.16	16 53.04	66.00	3 01.51	0.305
Sat.	3	2 38 39.03	9.574	15 28 31.8	44.54	15 52.81	66.09	3 08.55	0.282
SUN.	4	2 42 29.10	+ 9.598	15 46 13.1	+ 43.90	15 52 58	66.17	3 15.02	0.258
Mon.	5	2 46 19.74	9.622	16 03 38.8	43.24	15 52.35	66.25	3 20.91	0.234
Tues.	6	2 50 10.95	9.646	16 20 48.5	42.57	15 52.13	66.33	3 26.23	0.210

If the apparent time is ahead of mean time, add the equation of time to the mean chronometer time found; if behind mean time, subtract it. The chronometer's correction must also be applied, as in the longitude problem. This method should not be used in high latitudes.

SUMNER'S METHOD.

The following paragraph is taken from Bowditch, Art. 314: "By the various methods which have heretofore been described, in which single observations are alone concerned, the latitude has been found which places the ship upon a certain east and west

Page from the Nautical Almanac giving right ascension, declination, etc., of the sun corresponding to Greenwich mean noon.

AT GREENWICH MEAN NOON.

Day of the Week.	Day of the Month.	THE SUN'S				Equation of Time to be Added to Mean Time.	Diff. for 1 Hour.	Sidereal Time or Right Ascension of Mean Sun.
		Apparent Right Ascension.	Diff. for 1 Hour.	Apparent Declination.	Diff. for 1 hour.			
		h m s	s	° ′ ″	′	m s	s	h m s
Thur.	1	2 31 01.04	+ 9.529	N.14 52 26.2	+ 45.78	2 53.94	+ 0.328	2 33 54 98
Frid.	2	2 34 50.01	9.552	15 10 37.7	45.16	3 01.53	0.305	2 37 51.54
Sat.	3	2 38 39.53	9.575	15 28 34.2	44.54	3 08.56	0.282	2 41 48.09
SUN.	4	2 42 29.62	+ 9.599	15 46 15.5	+ 43.90	3 15.03	+ 0.258	2 45 44.65
Mon.	5	2 46 20.28	9.623	16 03 41.2	43.24	3 20.92	0.234	2 49 41.20
Tues.	6	2 50 11.51	9.647	16 20 50.9	42.57	3 26.24	0.210	2 53 37.75

line; by others the longitude has been found which places the ship upon a certain north and south line; but in neither case can the ship be located upon any particular point of the line. For instance, by the A.M. observation when the sun is near the prime vertical, the hour angle is determined with a certain degree of accuracy depending on the latitude by account, from which the

longitude is ascertained; hence the position of the ship is approximately known upon a certain meridian. Again, by the meridian observation, when possible to obtain it, the position of the ship is known upon a certain parallel of latitude; and by a combination of the two observations the ship's position may be located upon the chart; but there has been involved a lapse of time, which under some circumstances may be vital for the safety of the ship.

"But by the Sumner method the ship is at once located upon a line, generally oblique to both parallels and meridians; and in approaching a dangerous coast the importance of this line is obvious, since it exhibits on the chart the bearing of the land." There are two distinct cases for this method; namely, by the altitudes of two objects taken at the same time; for example, two stars as near 90° apart as possible; and by altitudes of the same object, observed at different times; this is perhaps the common case, the sun being the object observed. The second case only will be given.

Stated briefly, Sumner's method consists of working up a time sight of the sun, using two assumed latitudes, one larger and the other smaller than the latitude by account. This is worked up in the same manner as in working the longitude problem, and as we have used two different latitudes we get two different positions for the ship; plot these two points on the chart and connect them by a straight line, the ship is somewhere on that line. After a lapse of about 1½ hours get another altitude of the sun, note the course steered and the distance run in the interval between sights. Work this new sight up, using the same two latitudes as before, plot the points found, thus obtaining a new line of position for the ship. Now move the first line of position parallel to itself a distance equal to the run in the lapse of time, between the two sights, on the course made good. The intersection between the second line of position and the first line moved as directed is the position of the ship at the time of second observation.

Bowditch, Art. 318: "At sea, June 27, 1879, A.M., observed the following altitudes of the sun to determine the ship's place by the intersection of the lines of position; the ship having run in the interval 25 miles on a course E true.

Ship's place by account first observation Lat. 49° 40′ N. Long., 5° 53′ 30″ W.

Ship's place by account second observation Lat. 49° 40′ N. Long., 5° 15′ W.

Chro. time, 8h. 24m. 14s. Equ. of time (1st) + 2m. 40s.

Chro. time, 10h. 51m. 40s. Equ. of time (2d) + 2m. 41.3s.

⊙'s declination, 23° 20′ 55″ N.

⊙'s true alt. (1st) 36° 24′ 20″. ⊙'s true alt. (2d) 58° 9′ 20″

Assumed latitudes, 49° 50′ and 49° 30′.

Alt.	36° 24′ 20″		
Lat.	49 50 0	log sec	0.19043
P.D.	66 39 5	log cosec	0.03711
	2)152 53 25		
½ Sum (S)	76 26 42	log cos	9.36991
	36 24 20		
½ S–Alt.	40° 2′ 22″	log sin	9.80842
			2)19.40587
Local app. time, 7h. 57m. 34s.		log sin ½ t	9.70293 (Table 44)
Equ. of time	+ 2 40		
Local mean time, 8h. 0m. 14s.			
G'nw'h " "	8 24 14		
Long.	0h. 24m. 0s.		
Long.	6° 0′ 0″ W. (A)		

Alt.	36° 24′ 20″		
Lat.	49 30 0	log sec	0.18746
P. O.	66 39 5	log cosec	0.03711
Sum	152 33 25		
½ Sum	76 16 42	log cos	9.37512
	36 24 20		
½ S – Alt	39 52 22	log sin	9.80692
			2)19.40661
Local app. time, 7h. 57m. 20.7s.		log sin ½ t	9.70330
Eq. time	2 40.0		
Local mean time	8 0 0.7		
G'nw'h " "	8 24 14.0		
Long.	0h. 24m. 13.3s.		
Long.	6° 3′ 20″ W. (A_2)		

We can now plot these two positions, namely:
Lat. 49° 50 N, Long. 6° 0′ 0″ W, and Lat. 49° 30′ N, Long. 6° 8′ 20″ W, as shown by Fig. 21 at A_1 and A_2.

Alt.	58°	9′	20″		
Lat.	49	50	0	log sec	0.19043
P. D.	66	39	5	log cosec	0.03711
Sum	174	38	25		
½ sum	87	19	12	log cos	8.66985
½ S–Alt.	29	9	52	log sin	9.68781
					2) 18.58520
Local app. time	10h.	29m.	30.2s.	log sin ½ t	9.29260
Eq. of time		2	41.3		
Local mean time	10h.	32m.	11.5s		
G'nw'h " "	10	51	40		
Long.		19	28.5		
Long.	4°	52′	7″ W (B)		

Alt.	58°	9′	20″		
Lat.	49	30	0	log sec	0.18746
P. D.	66	39	5	log cosec	0.03711
Sum	174	18	25		
½ Sum	87	9	12	log cos	8.69603
½ S–Alt.	28	59	52	log sin	9.68554
					2)18.60614
Local app. time	10h.	27m.	15.8s.	log sin ½ t	9.30307
Eq. time		+2	41.3		
Local mean time	10	29	57.1		
G'nw'h " "	10	51	40		
Long.		0h.	21m.	43s.	
Long.		5°	25′	45″ W (B_2)	

Plot these two positions, namely:

Lat. 49° 50′ 0″ N Long. 4° 52′ 7″ W
Lat. 49° 30′ 0″ N Long. 5° 25′ 45″ W

as shown by fig. 21 B, B_2. Move A, A_2 due east (the true course made good) parallel to itself 25 miles and draw it as shown

by the dotted line A, A_2. Point of intersection O of dotted A, A_2 and B, B_2 is the position of the ship at time of second observation.

The ship position at O and the latitude and longitude of the spot is found as explained under the discussion of charts, etc.

TIDES.

The tide is the vertical and horizontal movement of the sea. Ordinarily the word tide is applied merely to the vertical movement, while the horizontal movements are called tidal currents. The rising of the water is called flood tide, the falling, ebb tide. The maximum height is called high water, distinguished by HW, the minimum is called low water (LW). The difference in elevation between high and low water is called the range of the tide.

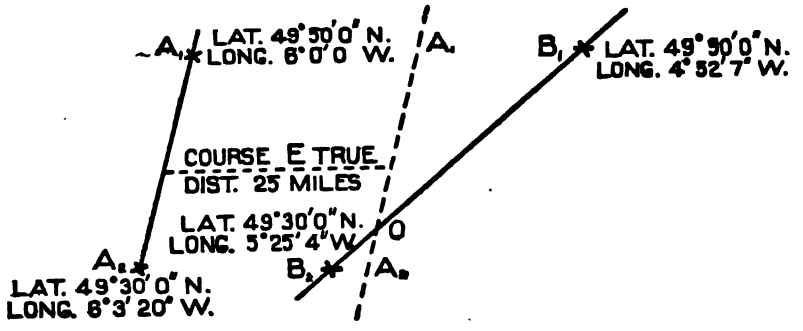


Fig. 21.

Theoretically every heavenly body causes a tide upon the earth, but the only tides considered are those caused by the sun and the moon. Tides are caused by the differences in the forces of attraction for different particles. The water on the portion of the earth under the moon is attracted by a greater force than that which attracts the earth as a whole. This force of attraction tends to elevate the surface at this portion of the earth. At the point opposite the moon, the water is attracted by a force which is less than the average, so that there is a tendency at this point also to elevate the surface. At points halfway between, the force tends to depress the surface. If the earth were set rotating, any place

would pass through two high and two low tides during one revolution, or one day.

If the earth, moon and sun were at rest this would explain what would occur, but the motions of these bodies, the varying depth of the ocean and the interference of land, make the actual result quite complicated.

The tides at any particular point depend chiefly upon local conditions, and cannot be exactly predicted, except from a long series of observations made at that point.

Effect of Changes in the Moon's Declination. When the moon is on the equator the two successive high tides would be of nearly the same height. When, however, the moon has a large north or south declination the two tides differ, as is shown in the figure.

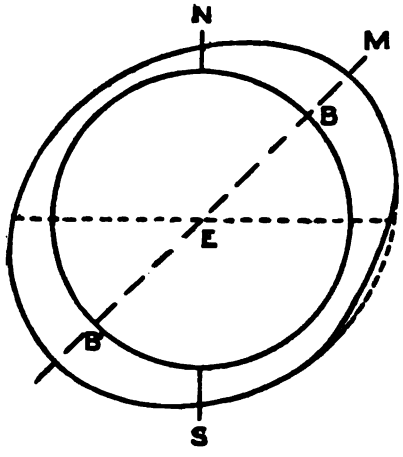
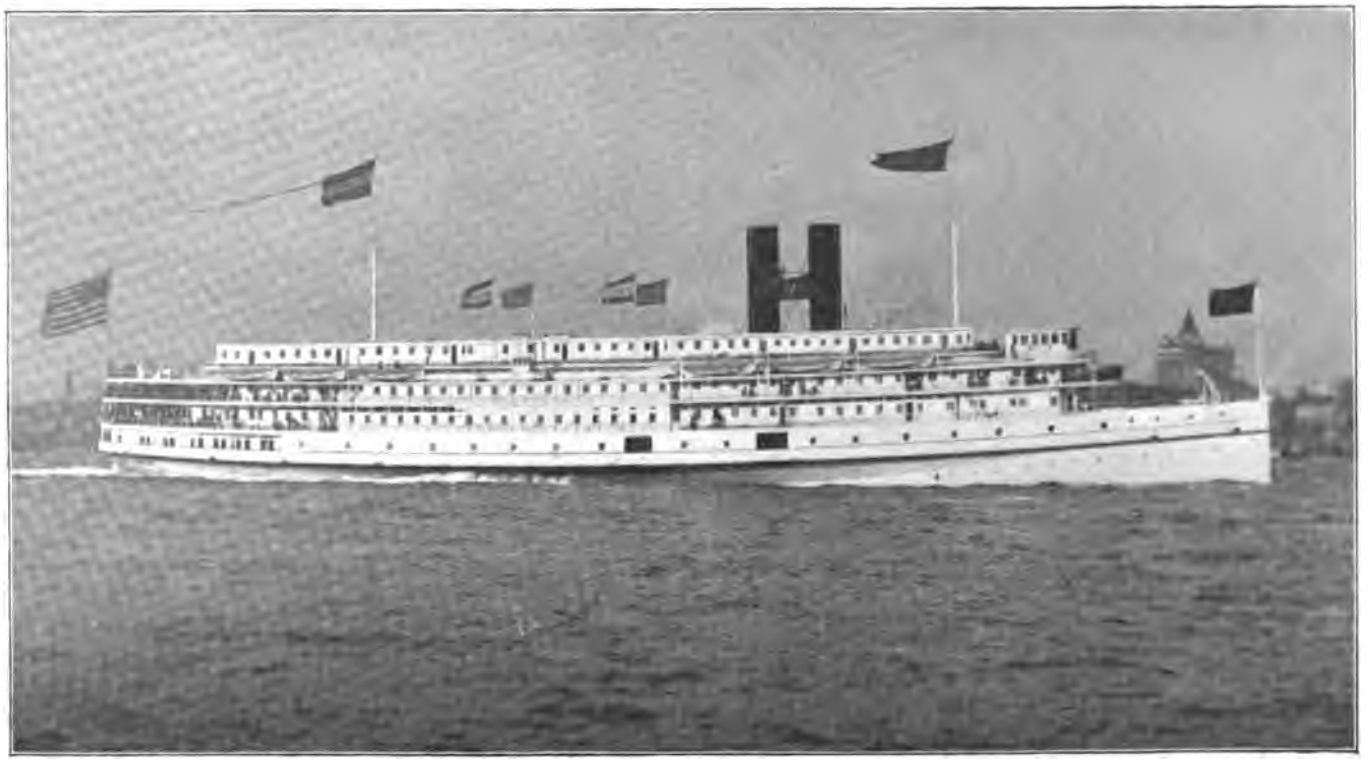


Fig. 22.

The moon attracts the earth with a greater force than it does the water on the opposite side, so that this part of the sea remains as it was. At a point B, under the moon, it is high water and the depth is greater than usual. Twelve hours later the same place will be at B, where it will be again high water, but not as high as usual. This is called the diurnal inequality. At points on the equator there is no inequality.

Effect of Moon's Distance. When the moon is nearest the earth (at perigee), the attraction is greater than the average and the tides are increased. When the moon is farthest from the earth (at apogee), they are decreased. Similarly, when the earth is nearest the sun (at perihelion), the attraction is increased. At aphelion it is decreased.

The force causing the solar tide is only a little more than one-third that which causes the lunar tide. At new and full moon, the sun, moon and earth are nearly in the same line, so that the solar and lunar tides combine and make the resulting tides higher



The finest type of sound steamer. From the designs of the Fore River Ship and Engine Company.

than usual. These are called *spring tides*. When the moon is 90° from the sun (at quadrature) the two tides tend to neutralize and the result is comparatively low tides. These are called *neap tides*.

On the days of new and full moon high water follows the moon's meridian passage by a certain interval of time depending upon the place. This interval is called the "establishment" of that port. At quadrature the high tides, although lower than usual, will follow by this same amount. Between these two positions the solar and lunar tides combine, so that the crest of the resulting wave is between the two. The result is that for a few days after new or full moon the crest of the wave is west of the lunar tide and high tide occurs earlier than the average. This is called the "priming" of the tide. A few days before new or full moon the crest is behind the lunar tide. This is called "lagging."

Effect of Wind and Change of Air Pressure. When the wind blows steadily into a bay or harbor the water is piled up and the height of tide increased. The time of high water is delayed; for the water continues to flow in after the predicted time has passed, so that the maximum does not occur until the effect of the wind and the ebb of the tide are balanced. When the pressure of the air is greater than usual the surface of the water is depressed below the normal. When the pressure is less, the height of the surface is greater. The amount of this change is usually small.

Compass Deviation by Azimuth of the Sun. The handiest method to use at sea in a sailing vessel, when an azimuth circle is not at hand and the apparent time at ship is fairly accurately known, and also the latitude, is to fix a shadow pin (not an iron one) erect over the center of the glass over the compass and get by its aid the compass bearing of the sun; then find from an azimuth table the true azimuth for the latitude, as explained in such tables. The compass bearing of the sun should be corrected to allow for the magnetic variation at the place, and the difference between the true azimuth and the corrected compass bearing is the deviation of the compass.

If an azimuth table is not at hand, the azimuth may be worked as indicated on pages 149–153 Bowditch, either by the method of time or altitude azimuths; the time azimuth is generally the best

method if the apparent time and latitude are fairly well known. This matter is not explained at length here, as such constant reference to Bowditch's tables have to be made. The navigator at sea has these tables, and in the portion preceding the tables an explanation of both time and altitude azimuths is given.

SECTION 4.

MECHANICS.

PART I.

Mechanics is that branch of science which treats of the action of forces and the effects which they produce.

Matter is anything which occupies space, and is made up of minute particles called *molecules*, which are in turn composed of atoms. The *molecule* is the smallest quantity of matter that can exist by itself. That is, if we were able to divide a molecule of a substance, the nature of the latter would be changed so that it would no longer have the same properties as before. While we do not know the size of the molecules, we have, however, some knowledge of their relative weights; that is to say we know how much a molecule of one substance weighs when compared with one of another. But if a molecule should be divided and the identity of the substance be lost, we should have, as a result of our division, particles still more minute called *atoms*, which may be defined as the smallest quantities of matter that can enter into combination. For instance, a molecule of water is composed of two atoms of hydrogen and one atom of oxygen, but any perceptible amount would be composed of myriads of molecules. There are some molecules which have a far greater number of atoms than water, such as a molecule of cane sugar for example, which contains 45 atoms. If a single one of these atoms should by any means be withdrawn from the molecule, the substance would no longer be sugar. It is probable that the smallest number of water molecules visible with the best microscope is 8,000,000. Atoms of various kinds combine to make molecules, and aggregations of molecules compose masses of bodies which may exist in three states called solid, liquid and aeriform; the last being divided into two classes, gases and vapors.

A solid is a body which retains a definite shape, *i. e.*, its molecules change their relative positions with difficulty.

A liquid is a body whose molecules cling together, yet easily change their relative positions, adapting themselves to the shape of the vessels containing them.

An aeriform body is one in which the molecules easily change their relative positions and tend to separate from each other to an almost indefinite extent; the air is the most familiar example.

It is probable that all bodies may exist in any one of the three states mentioned, just as water may exist as a solid (ice), as a liquid (water), and as a gas (steam). The term fluid is applied to liquids and aeriform bodies, and simply means that the substance to which it is applied is one in which the molecules easily change their relative positions.

The changes which may take place in substances are of two kinds, physical and chemical. A physical change does not affect the identity of the molecules, for example, water is always the same substance whether it is in the form of a solid, a liquid, or a gas. A chemical change does affect the identity of the molecule. For instance, water may by an electric current be changed into hydrogen and oxygen which are two gases having different properties.

A quality which belongs to matter, or is characteristic of it, is called a *property* of matter, and these are of two classes, physical and chemical. The physical are those which may occur without changing the identity of the molecule; the chemical those that can only take place by a change of that identity. Thus, for example, a piece of coal occupies space, it is hard, heavy and cannot move itself; properties exhibited by any piece of coal without any change taking place. They are the physical properties of coal. If, however, the coal is burned, we have a chemical change and the results of the combustion will not resemble coal. Hence, we may say that combustibility is a chemical property of coal.

What we have to deal with in this book are the physical properties of matter, and these may be divided into two classes: universal and characteristic. The former are possessed by all matter; the latter belong only to certain substances.

UNIVERSAL PROPERTIES.

Extensibility is that property of matter by virtue of which it occupies space.

Impenetrability is a property by virtue of which two bodies cannot occupy the same space at the same time, for it is evident that a piece of iron and a brick cannot occupy the same portion of space at the same time. Again, if an insoluble solid of any kind is thrust into a vessel containing water, the water rises, showing

that the solid occupies a space which was formerly occupied by the water.

Indestructibility indicates that matter can never be destroyed. It may be decomposed into other substances, as water into hydrogen and oxygen, but the number of atoms in the hydrogen and oxygen is the same as in the water, and there is not a single atom more or less than existed previously.

Inertia is the tendency which all bodies have to remain in their present state, whether that be one of rest or of motion. For example, a book placed upon a table cannot, of itself, leave the table, but must stay there until acted upon by some external force. If a stone is let fall it cannot arrest its own flight, but must continue in motion until stopped by the earth or some other body. If a coin is placed upon a card and the latter is suddenly moved to one side, the coin will fall, because its tendency was to retain its original position, so that the card moved from beneath it and it fell. Fly wheels continue to revolve after the motive power has ceased to act; cars tend to continue in motion after being started; people are thrown backwards by the sudden starting of a car in which they are standing because of the tendency of matter at rest to remain at rest. The reverse action takes place, if, after the car has attained considerable velocity, it is suddenly stopped, the passengers then being thrown forward by their tendency to continue in motion.

Mobility is that property in accordance with which the position of a body may be changed, as, for example, the book upon the table may be moved from that position to any other.

Divisibility is the property which indicates that matter may be separated into parts. Theoretically the atom is the limit of divisibility, but a practical limit is reached long before a separation into the molecules is attained. An instance of an exceedingly minute division is found in the case of coins which gradually lose in weight by use; yet the amount of matter lost with each transfer is beyond perception.

Porosity is the property which indicates that spaces may exist between the molecules. If a piece of iron is heated it expands, that is, it increases in size, simply because the molecules are farther apart and not because there is any more iron or any increase

in the number of molecules. Hence, we know that there must be greater space between the molecules of the body, even though we cannot see them. In the case of a sponge, we have sensible pores, an example, however, that cannot properly be included under this head.

Compressibility is the property in accordance with which matter may be reduced in size, or the molecules be crowded together so that they occupy less space.

Expansibility is that property of matter which indicates that it may be increased in size. It is the reverse of compressibility. For instance, if we compress a given bulk of air into a smaller space, we have illustrated its property of compressibility; while, if we remove the compressing force, it will at once expand, and thus manifest its property of expansibility.

Elasticity is the property by virtue of which bodies are able to resume their original form or size, after being changed by some external force. This is a property that is possessed to a certain degree by all substances, since there is universally a tendency to return to their original dimensions after a removal of the distorting pressure. It may be added that the amount of pressure does not alter the manifestation of elasticity except in the case of solids, since all fluids are perfectly elastic.

CHARACTERISTIC PROPERTIES.

Hardness is that property of matter by virtue of which some bodies resist an attempt to force a passage between their particles. It does not, however, imply that a body is heavier for lead is very heavy but is not hard. The diamond is much harder than gold, though gold weighs four times as much. The usual measure of hardness is the degree of difficulty with which it can be scratched by another substance. Fluids have no hardness.

Tenacity is the property which enables some bodies to resist a force tending to pull their particles asunder. For instance, it requires a considerable amount of force to break a wire, because the substance of which the wire is made is tenacious.

Brittleness is the property of matter which allows some bodies to be easily broken, as by a blow. A familiar example is found in glass.

Malleability is the property in accordance with which some bodies may be rolled or hammered into sheets. Steel has been rolled into sheets thinner than the paper upon which this is printed, and gold has been beaten so thin that 282,000 sheets placed one upon another would be but a single inch in height.

Ductility is the property of matter by virtue of which some bodies may be drawn into wire. Platinum has been drawn into wire $\frac{1}{3000}$ of an inch in diameter, and glass, which is ordinarily not easily ductile is very much so when heated to redness.

MOTION AND VELOCITY.

We now come to a consideration of *motion* and *velocity*. It may seem simple to define the absolute motion of a body, that is, to state just how far it travels with reference to some fixed point, but there is no point which we know to be fixed. All bodies are in motion, even the sun itself. Hence, in speaking of motion it must always be with reference to some point which is considered to be at rest.

Suppose an electric car is leaving a town at the rate of ten miles an hour. If two men sit in either end of the car, the distance between them does not vary; nor do their positions change with respect to the car. So, there is no relative motion between them; yet they are leaving the town at the rate of ten miles an hour. Therefore, both are in motion relatively to the town. Again, suppose a steamer to be sailing away from the land at the rate of 12 miles an hour and a man runs along the deck from stem to stern at the rate of 6 miles an hour. Here the ship has a motion relatively to the land; and the man has a different motion relatively to the same because he does not travel at exactly the same rate as the ship. If on the other hand the ship moved but 6 miles an hour and he was running at the same rate, there would, for the time being, be no relative motion between the man and the land, but there would be a motion of the ship relative to the land.

Velocity is the rate of motion, and is measured by the number of units of space passed over in a unit of time. If the same number of units of space are passed over in each succeeding unit

of time, the velocity is said to be *uniform* or constant, while in all other cases it is variable; a topic to be treated later.

In the example given above, a mile was used as the unit of space and an hour as the unit of time; the velocity of the car leaving the town being 10 miles, that of the steamer with reference to the shore 12 miles and that of the man with reference to both steamer and shore, 6 miles per hour. Feet per second is, however, the more usual basis in engineering calculations.

Let us now observe the difference between uniform and variable velocities. If a railroad train runs steadily at the rate of 40 miles per hour, its velocity is said to be constant; but if it first runs 30, then 40, and then 50 miles per hour its velocity will be variable. Stopping at stations of course renders its velocity variable. If the velocity of a railway train travelling 40 miles an hour is to be expressed in feet per minute, the number of miles is to be multiplied by 5280 to reduce them to feet, and the result divided by 60 the quotient being the number of feet, or 3520 per minute. This becomes 3520 ÷ 60 or 58.67 — feet per second. Velocity may be expressed by a formula where V represents velocity, S the space passed over, and T the time.

$$\text{Then } V = \frac{S}{T}$$

If S is in feet and T in seconds, the velocity is given in feet per second: if S is in miles and T in hours, the velocity is in miles per hour and so on. By transposing, the formula may be written in two other ways:

$$S = V\,T \text{ and } T = \frac{S}{V}$$

This shows us that by knowing any two of the three (space, time or velocity), the third may be obtained. These formulas can be stated in the following rules:

To obtain the velocity, *divide the space by the time.*

To obtain the space, *multiply the velocity by the time.*

To obtain the time, *divide the space by the velocity.*

EXAMPLES FOR PRACTICE.

1. Find the velocity in miles per hour of a train which travels 30 feet per second. Ans. $20\frac{5}{11}$ miles.

2. A body moves uniformly. The space traversed is 9000 feet, and the time is 4 seconds. What is the velocity in feet per minute? Ans. 135,000 feet.

3. The velocity of a point upon the rim of fly-wheel is 150 feet per second. How far will it travel in $1\frac{1}{2}$ minutes?

Ans. 13,500feet.

4. A steamer runs 455 miles at the rate of 14 miles per hour. How much time did it take to cover the distance?

Ans. $32\frac{1}{2}$ hours.

MOMENTUM.

The Momentum of a body may be defined as its quantity of motion. This is expressed by the product of the mass of a body and its velocity. If the mass of a body is M and its velocity V, the momentum is evidently M V. In a given time twice as much motion will be produced by two units of force as by one unit. This doubled momentum may exist in two units of mass having a unit of velocity or in one unit of mass with two units of velocity. For example: The momentum of a mass of 10 pounds having a velocity of 5 feet per second is the same as that of a mass of 5 pounds having a velocity of 10 feet per second. In either case the momentum is 50. Thus we see that momentum has reference only to force and inertia. From this fact we may deduce the statement that "Equal forces acting upon bodies free to move, will produce equal momenta."

EXAMPLES FOR PRACTICE.

1. What is the momentum of a ball weighing 500 pounds moving at the rate of 500 feet per second? Ans. 250,000.

2. Two bodies weighing respectively 50 and 70 pounds have equal momenta. The first has a velocity of 140 feet per second; what is the velocity of the other? Ans. 100 feet per second.

3. A piston weighs 500 pounds and has a momentum of 20,000; what is its velocity in feet per second?

Ans. 40 feet per second.

FORCE.

Force signifies any cause which tends to produce, to change, or to destroy motion. Various kinds of force will be taken up

later. Inertia shows us that bodies are incapable of changing their condition of rest or motion. Any cause capable of producing a tendency to change either of these conditions is called a force. The tendency of a force acting upon a body is to move it. When a force acts upon a body free to move, motion will be produced. If an equal force acts upon a body in the opposite direction the body will remain at rest. If we have a body with a spring balance attached and pull until the balance registers 5 pounds, we exert a force of 5 pounds on the body. In treating of forces we must consider three things:

1. *The point of application*, *i. e.*, the point at which the force acts.

2. *The direction*, or the line along which a force tends to move the point of application.

3. *The magnitude* or value compared with a given standard, or the relative rate at which it is able to produce motion in a certain body free to move.

In the above example of the spring balance, the point at which the balance is attached is called the point of application. The direction in which the balance is pulled is the *direction* of the force, and the 5 pounds which it registers is the *magnitude* of the force. If the resistance offered to the spring balance were not sufficiently great, motion would take place in the body to which it is attached.

The following propositions, which are of fundamental importance, are known as Newton's Laws of Motion.

Every body continues in its state of rest or of uniform motion in a straight line, unless compelled to change that state of rest or motion by the action of an external force.

Every motion, or change of motion, is in the direction of the force acting and is proportional to that force.

To every action there is always an equal and opposite reaction, i. e., action and reaction are equal and opposite in direction.

The first law is sometimes called the Law of Inertia of which it is a direct result. Perfect examples of the first cannot be given because all things within our reach or observation are acted upon by some external force. It means that if a body is once set in

motion it will move in the direction in which it is started and with a uniform velocity, unless some external force acts upon it, thereby compelling a change. If a bullet is fired from a rifle it travels in the direction in which it is fired, but the attraction of the earth for the bullet causes it to move toward the earth at the same time thus making its path a curve. Moreover the resistance of the air tends to decrease the velocity in the direction the bullet travels.

Inertia is a much misused term. It is not force although it is often so called. When a force acts upon a body and thus produces motion, the effect of the force is retained by the body and if the body is stopped, an equal amount of force to that which has been imparted must be used to overcome the motion. If a person jumps from a moving car his body has the same velocity as the car and force must be exerted in the opposite direction (by the earth) to overcome that velocity. A body also resists being put in motion, and, as we have just seen, resists being brought to rest when it is in motion. This is the resistance due to inertia, often erroneously called the resistance of inertia.

The second law is sometimes stated as follows: A given force will produce the same effect whether the body on which it acts is in motion or at rest; whether it is acted upon by that force alone or by others at the same time. This law shows that if two or more forces act upon a body, the effect which they produce will be proportional to their magnitudes and to the directions in which they act. As we have already seen, the bullet fired from a rifle travels in the direction in which it is fired and also falls toward the ground at the same time under the influence of the earth's attraction. It approaches the ground at the same rate that it would if simply dropped.

The third law states that action and reaction are equal and opposite, and is illustrated by the three following examples: A body weighing 2 pounds presses down upon a table with a force of 2 pounds when placed upon it. The table exerts a force of 2 pounds in the opposite direction. If a man pulls on a rope attached to a wall with a force of 100 pounds, the rope and wall oppose the pull which he exerts upon it with that same force, 100 pounds. A person jumps from a boat to the shore. To the force exerted by him in jumping ashore, there is an equal and opposite

reaction causing the boat to sink more deeply in the water and to go in a direction opposite to that in which the man jumped.

It is often convenient to represent forces by lines, one end of a line determining the point of application of the force, the direction of the line showing the direction and the length the magnitude of the same. For example: let AB be a straight line. The point A represents the point of application of the force and the direction in which the line is drawn shows the direction in which the force acts. Suppose now the line to be drawn to a scale of 10 pounds to the inch and that it is 6 inches long, the magnitude of the force represented would be 60 pounds. By drawing other lines to the same scale two forces acting at right angles to each other at the same point can be represented. These may be indicated by the lines AB and AC. Fig. 1. If AB is 4 inches long, and AC, 7 inches long, then the force represented by AB will be 40 pounds and that represented by AC, 70 pounds. The arrow heads indicate the direction in which these forces are acting, that is, the force AB tends to lift the point A, while the line AC tends to carry the point A to the right. These two forces may, however, be replaced by a single force which will have the same effect. AB tries to lift the point A; AC tries to carry it to the right. The two acting on A will tend to move the point A both higher and to the right at the same time, that is, in the direction of a line drawn from A somewhere between the lines AB and AC. This force which would replace AB and AC, may be determined by drawing from B a line parallel to AC, from C one parallel to AB, and joining the intersection of these two lines with the point A, forming the line AD. This line AD represents the magnitude and direction of a force acting on A which would be exactly equivalent in effect to the forces AB and AC. By measuring this line the magnitude of this force may be directly obtained. It may also be calculated from the lengths of the other lines. AB we know to be 4, and BD = AC = 7; therefore the line AD = $\sqrt{4^2+7^2}=\sqrt{65}$ = 8.06 +. It will be seen a rectangle has been formed in obtain-

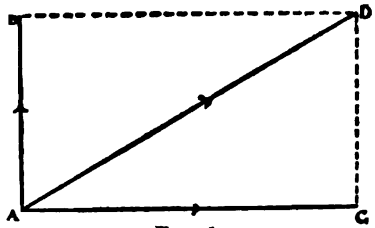


FIG. 1.

ing the force AD, which is called the *resultant* of AB and AC, and this figure is called a *rectangle of forces*.

It is not necessary, however, to have the forces at right angles in order to obtain their resultant, for if AB and AC, Fig. 2, act upon the point A, tending to move it in the directions of B and C, and the line BD is drawn parallel to AC, CD parallel to AB and the points A and D connected, then AD is the force which is exactly equivalent in its action to AB and A C combined, while its magnitude may be found as before, by measuring.

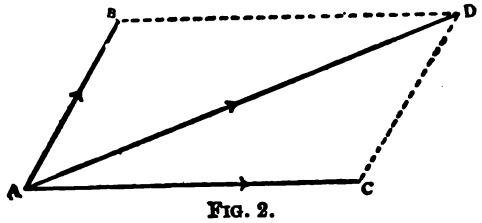


FIG. 2.

This is called the *parallelogram of forces*, and may be veri-

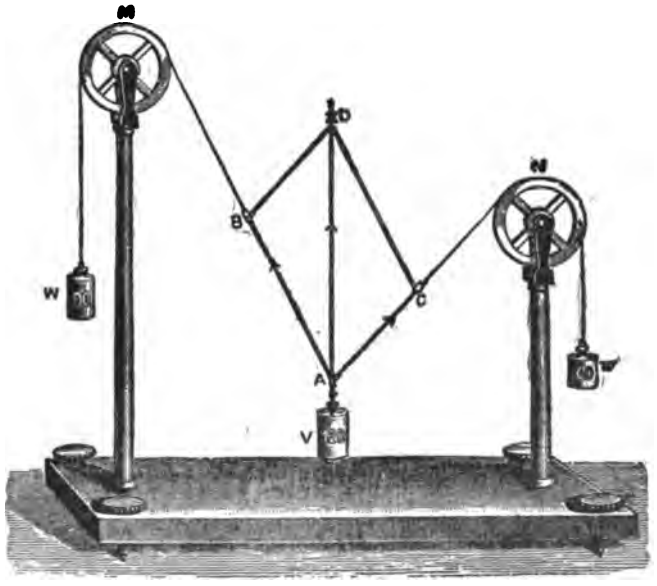


FIG. 3.

fied experimentally, as follows: suppose we have a stand with two uprights fitted with the pulleys M and N, Fig. 3. Let ABDC be a

light wooden frame, and let a weight of 120 ounces be hung at the point A. Attached to the point B is a cord which passes over the pulley M and has hung at its end a weight of 90 ounces. At the point C is similarly attached a cord passing over N with the weight of 60 ounces attached to it. The frame will then take the position shown in the figure, and AB, AC, and AD will be in the proportion of 90, 60, and 120. Hence we see that AB and AC might be replaced by a force AD of 120 ounces acting directly upward, and this we know to be exactly the amount necessary to maintain in equilibrium the weight of 120 ounces acting downward.

It will be seen from the above that the forces AB and AC are equivalent to a force AD acting directly upward. That is, they will exactly counterbalance a force equal to AD but acting downward, as the weight in Fig. 3. Furthermore, that a force equal to AD acting in the opposite direction will exactly counterbalance the forces AB and AC. When AD is drawn from the point A in direction opposite to that in which the forces AB and AC tend to move it the force AD is considered as acting in the direction of the pull of the weight. This force is called an equilibrant, and is the force necessary to maintain the forces AB and AC in equilibrium thus balancing their effects.

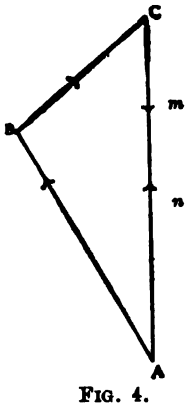


FIG. 4.

The forces AB and AC are called *components*, and AD is called the *resultant* or the *equilibrant* of these components according to the direction in which it is drawn. It should be observed that the following is always true for forces in equilibrium.

If a line is drawn from the point A, Fig. 4, representing the force AB, and is marked with an arrow showing that it acts in the direction AB, and from B the line BC is drawn proportional to and in the same direction as the force AC, and finally, from C the force CA equal to the force represented by DA, Fig. 3, we find that we have arrived at the point from which we have started. If the arrow is placed on CA so that it shall be represented as acting in the direction of the equilibrant (as shown at *m*) it will be seen

that the arrows follow each other around the sides back to the point from which the start is made. This shows that when forces drawn in this way form a triangle, or in fact any polygon, that this particular set of forces is in equilibrium. If the arrow of CA is turned in the opposite direction (as at n) the force represented will be equal in effect to AB and BC, and will be the resultant of the two.

Two forces of magnitude 12 and 5 act upon a point at right angles to each other. Find the magnitude of their resultant. Explanation: The resultant may be graphically represented as the third side of a right triangle whose sides adjacent to the right angle are 12 and 5. Let x be the value of the third side thus:

$$x^2 = 12^2 + 5^2 = 144 + 25$$
$$x^2 = 169$$
$$x = \sqrt{169} = 13 = \text{the resultant.}$$

EXAMPLES FOR PRACTICE.

1. A body receives an impetus from a force of 10 pounds acting toward the east. At the same time it receives an impetus from a force of 8 pounds acting toward the north. What is the resulting impetus? Ans. 12.8 + pounds.

2. Forces of 10 and 18 act upon a point, in northeasterly and easterly direction respectively. The angle between their line of action is 45°. Draw their resultant, giving direction and magnitude.

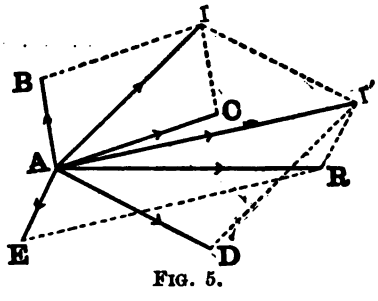


FIG. 5.

If there are more than two forces it is still possible to resolve them into a single resultant. For instance; suppose the forces AB, AC, AD and AE, Fig. 5, acting in the directions indicated by the arrows. To resolve these into a single component, we resolve the two forces AB and AC into their resultant Ar; then combine this resultant with the force AD to form the resultant Ar', and finally combine this force Ar' with the force AE obtaining their resultant which, as there are no more forces to take into consideration, is the final resultant or the force AR.

This resultant might have been more easily obtained by a polygon of forces, as follows: Starting from the point A, Fig. 6, draw the line AB representing the force AB in length and direction; from B the line BC equalling AC in magnitude and direction; in like manner draw the force CD equal to AD and from D the force DE equal to AE. The two points A and E are thus obtained, and are connected to obtain the magnitude and direction of a force equivalent to all of the forces, *i. e.*, the equivalent of AR acting at the point A. Its direction is indicated by the arrow which points in the opposite way around the polygon to that of the other arrows. If it pointed around in the same direction it would be the equilibrant, or force necessary to hold in equilibrium those which have thus been just decomposed.

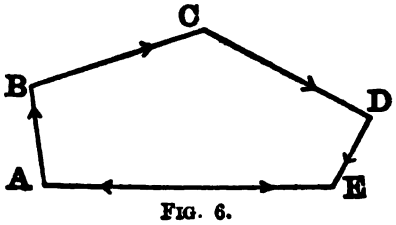


FIG. 6.

For practice let the student draw the diagram illustrating the following:

Given the forces AB = 10, AC = 13, AD = 21, acting away from the point A. The angle between AB and AC is 45°. The angle between AC and AD is 60°. Draw a figure showing the direction and magnitude of their resultant.

Explanation. Draw AB = 10 in any direction. At A lay off an angle of 45° or $\frac{1}{2}$ 90°. Draw AC 1.3 times AB. Draw AD from point A at an angle of 60° = $\frac{2}{3}$ 90° with AC, and make it 2.1 times AB.

Find the resultant of AB and AC = A*r*

Find the resultant of Ar and AD = A*r'*

A*r'* is the desired resultant and its direction is represented by an arrow pointing from A.

EXAMPLE FOR PRACTICE.

A force of 10 acts upon a point in a northerly direction and one of 16 in an easterly direction, the angle between their lines of action being thus 90 degrees. Draw a figure showing the direction and magnitude of their resultant. Ans. 18.8+

Forces not in the same plane may act upon a point. Thus, take the forces AB, AC and AD, Fig. 7, acting on the point A. If these forces are to be replaced by a resultant having an equivalent effect, the resultant of AB and AC is to be found first. This may easily be done as they lie in the same plane, by drawing B*r* and C*r*. Then the force A*r* is the resultant of AB and AC. But A*r* and AD lie in the same plane, as is readily seen by the figure, and their resultant may be determined by drawing the line *r*R and DR parallel to AD and A*r* respectively and connecting their point of intersection R with the point A, thus giving the desired resultant.

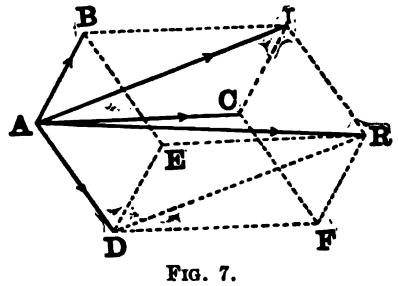


FIG. 7.

The lines BE, DE, ER, RF and FD complete a geometrical figure which is called a parallelopiped, the forces AB, AC and AD forming edges running to one of the vertices, and the resultant AR connecting the vertex A with the opposite vertex R. This is called a parallelopiped of forces.

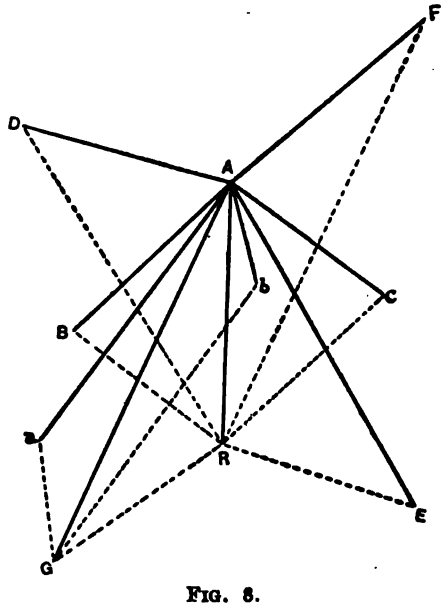


FIG. 8.

The Resolution of Forces is the operation of finding the components to which a given force is equivalent. As may be seen in Figure 8, it is possible to replace the force AR by the forces AB and AC, or by the forces AD and AE, or AF and AG; in fact it may be replaced in an infinite number of ways. Furthermore, any one or both of the two components shown in the figure may be replaced by two other forces, as shown in the case of the force

AG. The line AG may thus be replaced by the forces A*a*, and A*b*; hence, a force may be replaced by any number of other forces, since any force may be resolved into others.

CENTER OF GRAVITY.

All bodies are attracted by the earth. This attraction may be opposed by a single force acting upwards, and this single force acting in opposition to the attraction of the earth, must always pass through a point at which the entire weight of the body may be considered as being concentrated. This point is called the *center of gravity.*

If a block of stone is suspended by a rope from a derrick the force which sustains it, represented by the rope, must pass through the center of gravity, for if this is not the case, the stone will tip until the sustaining force fulfills the condition named. If a body is moving, its path is considered to be along the line described by its center of gravity, and the distance that a body is moved is always taken as the distance which its center of gravity has moved.

The center of gravity of a homogeneous sphere, that is, one of the same constitution throughout, is at its geometrical center. The center of gravity of a round iron bar 10 feet long, is in the line passing through the middle, *i. e.*, the axis of the bar, and is 5 feet from either end. If the bar were larger at one end the center of gravity would be nearer that end.

In order to learn how to locate the center of gravity, take a triangular piece of paper, or any convenient shape, pass a pin through one corner, and from the pin hang a thread with a weight attached. Hold the paper and thread up by the pin. The attraction of gravity will cause each to hang vertically and the thread will cross the center of gravity of the paper, for the center of gravity must be in the same line as each is suspended from the same point. Draw a line on the paper with a pencil to mark the position of the thread relative to the paper. Take another corner of the paper and perform the same operation. The two lines thus drawn will intersect, and the center of gravity will be at the point of intersection, because, in each case, it was under the line indicated by the thread.

The center of gravity is not necessarily inside the body. In the case of a convex ring it is at the center, and in the case of an angle iron it is somewhere between the two legs. If a body of this kind were to fall, the center of gravity would proceed in a straight line irrespective of the curves which might be traced by any of the other parts.

If the center of gravity is supported, the whole body is said to be in a state of equilibrium. If it is supported in such a way that, when slightly displaced from its position of equilibrium, it tends to return to that position, it is said to be in *stable* equilibrium. This is true since we raise its center of gravity if we displace the body, and the center of gravity tends to resume its former position. If a body is placed in such a way that when slightly displaced from its position of equilibrium it tends to fall still further from that position, it is said to be in *unstable* equilibrium. In such a case the displacement lowers the center of gravity and the body will not come to rest until the center of gravity has assumed the lowest possible position.

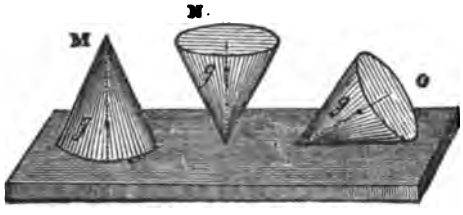


FIG. 9.

A body is in *neutral* equilibrium if it is supported in such a way that, when displaced from its position of equilibrium, it tends neither to return, nor to fall further from it. Such a displacement neither raises nor lowers the center of gravity.

Fig. 9 illustrates the three states of equilibrium by the three cones, the center of gravity being marked in each case.

In the first case, M, if the cone is tipped the center of gravity is raised and the cone tends to resume its former position.

In the second case, N, if the cone is tipped a little it will tend to fall further from its position, because the center of gravity will tend to assume the lowest position which it can attain.

In the third case, O, if we move the cone by rolling it the center of gravity is neither raised nor lowered, and the cone is therefore in neutral equilibrium.

A vertical line drawn downward from the center of gravity is called the line of direction. It represents the direction in which the center of gravity would move if the body were not supported. It may be considered as a line connecting the center of gravity of a given body with the center of the earth. The part on which a body rests is called its *base*. If a body is supported on legs, as a table, the base is the polygon formed by joining these points of support. If the line of direction falls within the base the body stands, but if it falls outside the base the body falls. The farther within the edge of the base the line falls, the more stable is its equilibrium, and this stability of bodies is measured by the amount of work necessary to overturn them. This depends upon the distance that it is necessary to raise the center of gravity, which is equivalent to raising the whole body, in order to make the line of direction fall without the base. In the second case, that of the cone resting on its apex, the slightest movement carries the center of gravity outside the base and the cone falls, and it has not been necessary to raise the center of gravity at all. In the preceding case, however, it would be necessary to raise the center of gravity in order to tip it over.

We have said that the center of gravity is that point at which the whole weight of a body may be considered as concentrated. It is often convenient to locate the center of gravity of various systems and shapes of bodies. To find the center of gravity of two bodies whose distance apart is known, we divide the line connecting their centers of gravity into two parts which are inversely proportional to their respective weights. Suppose for example that it is desired to find the center of gravity of two bodies, A and B, Fig. 10, weighing 10 and 15 pounds respectively, the distance between whose centers of weight is 20 feet. Let x represent the distance of the common center of weight from the center of the larger body, and $20 - x$, the distance from the center of the smaller. Then

$$\begin{aligned} 10 : 15 &= x : (20 - x) \\ 15\,x &= 200 - 10\,x \\ 25\,x &= 200 \\ x &= 8 \end{aligned}$$

Hence the common center of weight is at G a distance of 8 feet from the larger and 20 — 8 = 12 feet from the smaller body.

Now let us consider a system of three bodies lying in the same plane whose common center of gravity we wish to find. First find the center of gravity of two of the bodies, as explained.

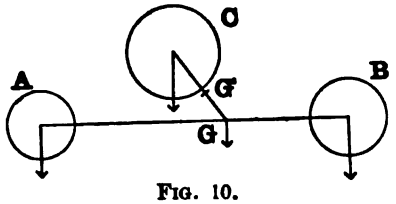


FIG. 10.

This center of gravity may be considered as the point at which the weight of the two bodies is concentrated. It is then merely necessary to find a second center of gravity, using as the weights the combined weight of the first two and the weight of the third.

Suppose that there is a third weight C of 20 pounds in the same plane with the weights considered in the previous example, and at a distance of 5 feet from their common center of gravity. In order to find the center of gravity of the system, we have merely to find the center of gravity of two weights of 25 and 20 pounds at a distance from each other of five feet. Then,

$$20 : 25 = x : 5 - x$$
$$25x = 100 - 20x$$
$$45x = 100$$
$$x = 2\tfrac{2}{9}$$

Therefore, the center of gravity of the system is at G′ on the line connecting the third weight with the center of gravity of the first two, at a distance of $2\frac{2}{9}$ feet from that center.

This method may be followed in determining the center of gravity of any number of bodies lying in the same plane. For plane figures the center of gravity is easily found. In the case of a triangle it lies on a line from any vertex to the middle of the opposite side at $\frac{2}{3}$ of the distance from the vertex to that side. The center of gravity of a parallelogram lies at the intersection of its diagonals. It will be found to coincide with the geometrical center of any regular plane figure.

EXAMPLES FOR PRACTICE.

1. The distance between the centers of gravity of two homogeneous bodies is 12 feet. The bodies weigh 20 and 30 pounds.

Where is their common center of gravity.

Ans. { On a line joining their centers. / 4.8 feet from larger body.

2. A, B and C weigh 4, 8, and 16 pounds respectively. They are connected by a rod which we will suppose to be without weight. The distance A B is 10 feet and B C is 24 feet. Find their common center of gravity.

Ans. 11$\frac{5}{7}$ feet from C, on part of the bar between B and C.

GRAVITATION.

Every particle of matter in the universe has an attraction for every other particle, and this attractive force is called gravitation. It acts instantaneously, while light and electricity require time to traverse space, so if a new star were to be created far off in space, its light might not reach the earth for hundreds or thousands of years and it would be invisible for many generations to come, but its pull would be felt by the earth on the instant, and would not be affected by the interposition of any other substance. For instance, the moon might intervene between the star and the earth, but the influence of the star would not be annulled. Gravitation is independent of the kind of matter, but depends upon the quantity or mass and the distance. Mass does not necessarily mean size. The planet Jupiter is about thirteen hundred times as large as the earth, but has only about three hundred times as much matter, because it is only about one quarter as dense.

Gravitation varies 1: directly as the product of the masses, and, 2: inversely as the square of the distance (between the centers of gravity). For example, doubling the product doubles the attraction, but by doubling the distance the attraction is reduced to one fourth. Suppose the relative weights of two masses to be 2 and 3 and the distance between them 10 units of measure. The product of their masses is 2 times 3 or 6, but if the first were to be doubled becoming 4 instead of 2, the attraction would be doubled because 4 times 3 are 12, and 12 is twice 6. If the distance were increased from 10 to 20, the attraction would be but one fourth as great, because 10 times 10 is 100, 20 times 20 is 400, and 100 is $\frac{1}{4}$ of 400. If the distance were decreased to 5 instead of 10, the at-

traction between the bodies would be 4 times as great since 10 times 10 is 100 which is 4 times as great as 5 times 5 or 25.

The force exerted by one body upon a second is the same as that exerted by the second upon the first. If a stone is falling to the ground, the stone has the same attraction for the earth as the earth has for the stone, each having the same momentum, but the mass of the earth is very great in comparison with that of the stone, and the earth's velocity is so infinitely small that we can have no conception of it. The above is the most familiar illustration of gravitation, that is: the attraction between the earth and bodies on or near its surface. This particular form of gravitation is commonly called *gravity*. The measure of the attraction is called the weight and its direction is that of the plumb line, or, vertical.

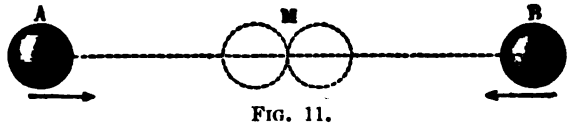


FIG. 11.

Suppose we have the two masses A and B (Fig. 11), which are exactly equal to each other and whose reciprocal attraction is therefore the same; then, unless there is some other force interfering with or destroying this attraction, it is clear that the two masses acting under the influence of their mutual attractions will

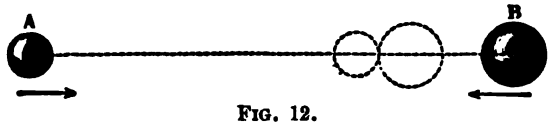


FIG. 12.

approach each other at the same speeds, until they come in contact at the point M, which is exactly at the central point of the distance originally separating them. Again, in Fig. 12 let us suppose the mass B to be twice as great as that of A, then, as they approach each other A will have twice the velocity of B, and it will consequently travel twice as far. We thus see that the smaller the mass the greater will be the distance that it must travel to reach the meeting point, and if this difference between the two masses be still further increased as in Fig. 13, where A is 1 and B 100, the distance traveled by A will be correspondingly increased. In this case the motion of B would be so small that it

would apparently remain at rest, while the smaller body A would move up to it with great rapidity. This, then, is the explanation of the phenomena which we see every day; namely, the fall of bodies, for in comparison with the earth, all bodies upon its surface are exceedingly small and are therefore drawn to it by a powerful attraction. Weight or gravity is, therefore, the reason

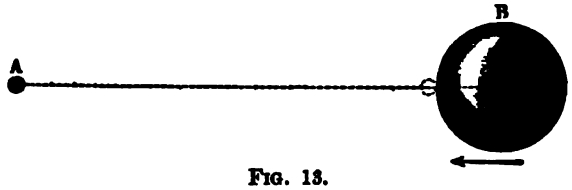


FIG. 13.

for the fall of bodies, and observation has shown that if the time during which a body is allowed to fall is one second the distance through which it will fall will be about 16.08 feet.

If we suspend a body, such as a plumb bob for example, by a cord so that it cannot fall, it will give a direction to the thread which will indicate the direction in which gravity is acting. We call this a vertical or plumb line, and is shown in its simplest construction in Fig. 14. The direction which the vertical cuts at right angles is called a horizontal line. The surface of quiet water is always horizontal.

FIG. 14.

The total attraction of the earth at E appears to be concentrated at its center C, as shown in Fig. 15. Every body upon its surface, which is distant from the central point of attraction by half the diameter of the earth, *c r*, will, therefore, be drawn toward it with a strength that will cause it to fall a distance of 16.08 feet in the first second. At greater distances the attraction is no longer the same, but becomes less the farther we go from the center of the earth. This loss of weight follows a certain definite law which may be thus expressed: If the force of gravity at a distance 1 from the center of the earth causes a body to fall 16.08 feet in a second, then at a distance of 2, it will cause it to fall $\frac{16.08}{4}$ feet; at a distance of 3, $\frac{16.08}{9}$ feet; at 4 it will fall $\frac{16.08}{16}$ feet, etc. Hence for any given distance the force of gravity can be indicated by a fraction whose numerator is 16.08 and whose denominator is

obtained by squaring the distance, or expressed more briefly: *the force of gravity decreases inversely as the square of the distance.*

We would naturally think, then, that upon very high mountains the distance fallen in a second would be somewhat less than 16.08 feet. But the highest mountains of the earth are but mere points in comparison with the whole, so they cannot exert an influence upon the rate of fall that can be observed.

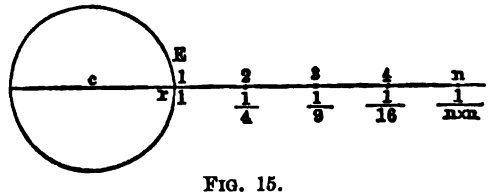


FIG. 15.

In considering the force with which the earth attracts various bodies, the absolute or fundamental unit is the one which by acting for a unit of time upon a unit of mass will produce a unit acceleration or gain of velocity.

In the English system of units the foot-pound-second unit of force is ordinarily used. That is, the unit of time is the second, the unit of mass, the pound, and the unit of gain of velocity (acceleration) 1 foot per second.

The unit of force is a poundal. Thus the poundal may be defined as the force necessary to produce an acceleration, or gain of velocity of 1 foot per second, when it acts on a mass of 1 pound for 1 second. If a poundal acted for 1 second on a mass of 1 pound, the mass of 1 pound would acquire a velocity of 1 foot per second supposing it to be at rest. If it was travelling at the rate of 1 foot per second and a force of 1 poundal acted upon it for 1 second, it would have a velocity of 2 feet per second, and so on. The earth's attraction for a pound is much greater than a poundal, varying according to position, whether near the equator or the poles, and being greater at the latter. If the attraction of the earth is allowed to act upon a body for 1 second, the body attains a velocity of about 32.16 feet per second. That is, the attraction of the earth is 32.16 poundals (at the latitude of New York). If the attraction of the earth is exerted upon a body free to fall for 2 seconds it acquires twice that velocity, and so on.

We have said that the attraction of one mass for another varies inversely with the square of the distance. It has been seen that, at the surface, the earth's attractive force for a pound mass

would be 32.16 poundals — for a mass of 2 pounds 64.32 poundals; for a mass of 3 pounds 96.48 poundals, and so on. On ascending from the surface for a considerable distance, we find that the law of gravitation as before formulated holds good; that is to say, the attraction of the earth decreases in proportion to the square of the distance. The mass of the earth, of course, remains constant, so that if we double the mass, the attraction of the earth for the same is doubled, from which the law of weight may be formulated as follows;

The weight of a body varies directly as the mass, and inversely as the square of the distance between its center of gravity and that of the earth.

This is for bodies above the surface of the earth, and means simply that if the mass of a body is doubled or trebled, the attraction of the earth for that body is doubled or trebled and that, if the distance between the centers of gravity of the two bodies is doubled, the attraction will be $\frac{1}{4}$ as great. If we treble the distance between the center of gravity of the earth and the body, the attraction will be $\frac{1}{9}$ as great, while, if the mass of a body is doubled it weighs twice as much; if it is trebled, three times as much, and so on.

A body at the earth's surface weighs 700 pounds; what would it weigh 4000 miles above the surface. The distance to the earth's center is 4000 miles. Ans. 175 pounds.

It is known that weights are inversely proportional to the squares of their distances from the earth's center. Let W represent the surface weight, D the surface distance, w its weight at an elevation and d the elevated distance from the earth's center.

Then, $w : W = D^2 : d^2$

or $w : 700 = 4000^2 : 8000^2$

Solving, we get

$$w = 175$$

EXAMPLES FOR PRACTICE.

1. Find the weight of a 180 pound ball, 2000 miles above the earth's surface. Ans. 80 pounds.

2. How far above the surface of the earth must 2700 pounds be placed to weigh 1200 pounds? Ans. 2000 miles.

Bodies weigh most at the surface of the earth. If we go below that surface they do not weigh as much. We have just seen that as we go away from the earth the weights of bodies vary inversely as the square of the distance. If we go below the surface, however, weight decreases according to a different law, namely: *It decreases as the distance to the center decreases.* That is, if the body is only half as far from the center as the surface is, it weighs but one-half as much. This is due to the fact that the mass between us and the center of the earth becomes less. Consequently the force of attraction is less.

Find the weight of a 180 pound ball 2000 miles below the earth's surface.

Using similar symbols as when finding the weight of a body above the earth's surface, our formula becomes from the statement above:

$$w : W = d : D$$

Then

$$w : 180 = 2000 : 4000$$
$$w = 90$$

EXAMPLES FOR PRACTICE.

1. What would a body weighing 550 pounds on the earth's surface weigh 1,000 miles below the surface? Ans. $412\frac{1}{2}$ pounds.

2. (*a*) Would a 50 pound ball weigh more 1,000 miles above the earth's surface, or 1,000 miles below it? (*b*) How much more?

Ans. { (*a*) Below. (*b*) $5\frac{1}{2}$ pounds.

FALLING BODIES.

A force acting continuously upon a body causes it to move faster and faster if it is free to move, as in the case of a stone falling to the earth.

If a small stone were to fall a few inches it would not produce any particular effect upon striking because it has not acquired

sufficient velocity, but if it were allowed to fall from a greater height, however, it might do considerable injury, as it would then have acquired a high velocity.

We have seen that the attraction of the earth was 32.16 poundals, so that, at the end of 1 second, the velocity of a body is 32.16 feet per second. At the end of the second second it would be twice this or 64.32 feet; at the end of the third second, three times this, or 96.48 feet. The symbol used to denote 32.16 is g, which is the value of gravity (in this case at the latitude of New York), and it does not vary greatly on the earth's surface. In order to obtain the velocity of a body after it has fallen for a certain length of time, g or 32.16 is multiplied by the number of seconds during which the body has been falling; or expressing it as a formula,

$$v = g\,t$$

in which v is the velocity, and t the number of seconds. So, if a body has been falling for 4 seconds, its velocity is $32.16 \times 4 = 128.64$.

In a fall of 1 second we know that a body attains a velocity of 32.16 feet. At the beginning of the second its velocity was 0. Hence its average velocity is $0 + 32.16 \div 2$ or 16.08. Hence a body free to fall under the influence of gravity, falls in the first second 16.08 feet. At the end of the second second its velocity is 64.32 feet per second. At the beginning it was 32.16. Therefore the space passed over in the second second is $\frac{32.16 + 64.32}{2}$ or 48.24 feet.

Again, it begins the third second with a velocity of 64.32 feet and ends it with a velocity of 96.48 feet per second. Hence its average velocity during the third second is $\frac{64.32 + 96.48}{2}$ or 80.4 feet per second.

The whole distance passed through by a body falling from a state of rest, may be calculated as follows:

The velocity at the beginning was 0. The final velocity was gt, that is, 32.16 times the number of seconds, and the average velocity is the sum of these divided by 2, or $\frac{gt}{2}$; hence, the number

of seconds it has fallen being t, the space may be calculated by multiplying the time t by the average velocity $\frac{gt}{2}$. Thus the formula for the total space passed over becomes $S = \frac{1}{2} gt \times t = \frac{1}{2} gt^2$. From the two formulas just derived, namely: $v = gt$ and $S = \frac{1}{2} gt^2$, it is possible to find, if the time is known, the velocity and the space passed through because g is already known.

TABLE I.

Number of seconds.	Feet fallen during each second.	Velocities at the end of each second.	Total number of feet fallen.
1	16.08	32.16	16.08
2	48.24	64.32	64.32
3	80.40	96.48	144.72
4	112.56	128.64	257.28

Considering 16.08 feet as one space

TABLE II.

Number of seconds.	Spaces fallen during each second.	Velocities at the end of each second.	Total number of spaces fallen.
1	1	2	1
2	3	4	4
3	5	6	9
4	7	8	16
t	2t–1	2t	t^2

In the above tables the initial space or the space traversed by a falling body during the first second is $\frac{g}{2}$ or 16.08 feet. Considering the second columns of our tables it will be seen that the space fallen in each second may be found by multiplying twice the number of seconds already fallen minus one, by one-half the acceleration

due to gravity. If we call the space fallen s, the time t, and the acceleration of gravity g, we may deduce the formula,

$$s = (2t - 1)\frac{g}{2}$$

The third column of the tables gives the velocities at the end of each second. It is known that the velocity at the end of the first second is twice the time multiplied by one-half g, or:

$$v = 2t \times \frac{g}{2}$$
$$= gt.$$

From the fourth column it will be seen that the total space traversed is dependent upon the square of the time or t^2. Therefore it is only necessary to multiply the space fallen in the first second, or $\frac{g}{2}$, by the square of the time; and denoting the total space by S, the formula

$$S = \frac{g}{2} t^2$$

is obtained.

If the body has a certain velocity at first, or, if it has fallen for a certain time, and its velocity is to be determined after a given amount of time has elapsed, its original velocity makes it possible to calculate the number of seconds during which it has already fallen, to which must be added the time given. This sum will be the total time, which multiplied by g will be the final velocity. If the space passed over is to be calculated from the time which is given; ascertain how far the body had fallen in the first place, and subtract that space from the total space passed over.

What is the velocity of a body that falls freely for 13 seconds? Solution. The formula for velocity is, $v = gt$. In this case the time is 13 seconds, and g is 32.16 feet. Hence the velocity is 32.16×13 or 418.08 feet per second. To find the space traversed in 13 seconds, substitute the proper values in the formula

$$S = \tfrac{1}{2} gt^2$$
$$= \tfrac{1}{2}\, 32.16 \times 13^2 \quad \text{Then the result is—}$$
$$= 2717.52 \text{ feet.}$$

EXAMPLES FOR PRACTICE.

1. A body falls freely for 10 seconds. (a) What is its velocity at the end of that time? (b) How far has it fallen?

Ans. (a) 321.6 feet per second.
(b) 1608 feet.

2. The space fallen is 1945.68 feet. (a) What is the time? (b) What is the velocity?

Ans. (a) 11 seconds.
(b) 353.76 feet per second.

3. The velocity at the end of a certain time is 385.92 feet per second. If the body has fallen from rest; (a) find the time; (b) find the space fallen.

Ans. (a) 12 seconds.
(b) 2315.52 feet.

PROJECTILES.

Projectiles are acted upon by three forces:—an impulsive force, such as the pressure generated by powder; the attraction of gravitation and the resistance of the air. The action of the last is complicated and will here be wholly neglected.

With regard to the second of the three, we have already seen that the *time* which any body takes in falling depends solely on its *height* above the surface of the earth. If it is 16.08 feet above the surface it takes one second to fall to the earth, and this time is irrespective of other motions such as twisting and horizontal motion, *i. e.*, of any motion which does not act to oppose or assist the force of gravity.

Suppose, in the following discussion, that the force of gravity is 32.16, which means that in the first second a body under the attraction of the earth will fall, 16.08 feet.

If a gun is placed at a height of 16.08 feet above the surface of the earth, and a projectile is fired from it horizontally, one second elapses before the projectile strikes the earth—supposing the ground to be level at the place. Hence, if the impulsive force of the gun imparts a velocity of 1800 feet per second to the

projectile, the horizontal distance travelled by it, since it is one second in the air before striking, is 1800 feet. A ball fired horizontally from the gun and one dropped from the muzzle at the same instant, reach the ground at the same time, the latter just below the muzzle of the gun, the former at a certain distance in front of it called the range, which point it reaches after describing a curve called a parabola. The former has travelled at a much faster rate in general, because it has gone farther, but it has travelled *towards the earth at the same rate* as the latter. If the gun were in a position four times as high, or 64.32 feet above the earth, and fired horizontally the projectile would be in the air 2 seconds; and if its velocity were 1800 feet per second, its range would be 3600 feet.

If the gun is fired horizontally, the range known to be 2700 feet, and that the velocity is 1800 feet per second it is possible to calculate the height of the gun, for the time is $\frac{2700}{1800} = 1\frac{1}{2}$, and in $1\frac{1}{2}$ seconds a body falls 36.18 feet—which is the height of the gun above the ground.

The range is increased by inclining the gun so that it points upward, the angle for greatest range depending in practice on the velocity imparted, the angle for maximum range in a vacuum being 45°.

EXAMPLES FOR PRACTICE.

1. A gun is 64.32 feet above the ground. The projectile attains a velocity of 2,000 feet per second. What is the range (in a vacuum) if the gun is fired horizontally?

Ans. 4,000 feet.

2. What is the range of a gun fired horizontally, the velocity of the projectile being 1,500 feet per second, if the gun is 4.02 feet above the ground? Ans. 750 feet

PENDULUM.

A simple pendulum is defined as a gravitating particle, supported by a line or cord without weight. This is evidently a theoretical pendulum, the compound or physical pendulum being a weight so suspended as to be capable of oscillating about a fixed point. Compound pendulums are of many forms. They are often

made of a steel rod, thin and flexible, and carrying at the bottom a heavy mass of metal known as a bob.

The motion from one extremity of the arc, a b, Fig. 16, to the other is called a *vibration* or *oscillation*. The time occupied in moving over this arc is called the *time of vibration*. The angle a f b measures the *amplitude of vibration*.

If the times of vibrations of pendulums with different lengths are considered, it will be found that the short pendulum vibrates more rapidly than the long one. It is evident, then, that the parts nearest the center of suspension tend to move faster than those further away. Between these there will be a particle moving of its own accord at the average rate of all the other particles. This particle must therefore move as if vibrating alone since it fulfills the conditions of the simple pendulum. This point is called the *center of oscillation*. The length of the simple pendulum is the distance between the centers of suspension and of oscillation of any compound pendulum.

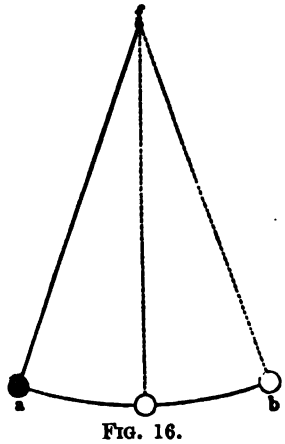


FIG. 16.

There are three important laws of the pendulum.

1. *The vibrations of a given pendulum at any given place are isochronous*, that is, they are performed in equal times.

2. *The time of vibration is independent of the weight or material of the pendulum.*

3. *The lengths of pendulums are directly proportional to the squares of the times of vibration, or inversely proportional to the squares of the numbers of vibrations in a given time.*

These laws are easily verified by experiment. It is usual to take as the arc of vibration, one not greater than 5 degrees, since the first law is true only when the pendulum vibrates in a cycloidal arc. The time of vibration is dependent upon the length of the pendulum and the intensity of the force of gravity at any given place; for it will be found that as g varies, so does the length of the second pendulum if we wish it to vibrate in the same time at two places. This statement may be expressed by the formula,—

$$t = \pi \sqrt{\frac{l}{g}}$$

where t is the time of vibration, l the length of the pendulum, and g is the acceleration due to gravity. An inspection of this formula will show that we have here a direct method for the determination of g.

From the third law we may deduce a method of determining the lengths of an equivalent simple pendulum if we have given the length and time of vibration of a compound pendulum. As the lengths are directly proportional to the squares of the times of vibration, we may write the proportion,—

$$1 : l = 1^2 : t^2,$$

where 1 represents the length and 1^2 the square of the time of vibration of a second pendulum; l is the length and t the time of vibration of the compound pendulum.

The pendulum finds practical use as a regulator for the motive power of clocks. Weight or springs tend to produce accelerated or retarded motion, but the pendulum keeps the motion uniform. If the clock gains time the pendulum is lengthened, if it loses, the pendulum is shortened.

Find the length of a pendulum which will beat half seconds. The length of the second pendulum is 39.1 inches. Solution—

$$L : l = 1^2 : t^2$$

where L is the length of the second pendulum, l the length of the pendulum to be found and t its time of vibration, substituting the above values we have:

$$l = 39.1 \times (\tfrac{1}{2})^2$$
$$l = 9.775 \text{ inches.} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE:

1. The length of the second pendulum being 39.1 inches, what is the length of a pendulum that vibrates 5 times in 4 seconds? Ans. 25.024 inches.

2. What is the length of a pendulum which beats once in 3 seconds? The length of the second pendulum is 39.1 inches. Ans. 351.9 inches.

What is the value of g where the length of the second pendulum is 39.1 inches? Solution. Use the formula

$$t \text{ (in this case 1)} = \pi \sqrt{\frac{l}{g}}$$

in which $g = 3.1416^2 \times l$, substituting for l its value, $g = 3.1416^2 \times 39.1 = 385.9 \times$ inches.

$g = 32.16$ feet (approx.) Ans.

1. What is the value of g where the length of the second pendulum is 39 inches? Ans. 32.08 feet (approx).

2. Find the value of g when the length of the second pendulum is 39.2 inches. Ans. 32.24 feet (approx).

ENERGY.

Work is the overcoming of resistance of any kind, so, when a force causes motion through space it does work. If no resistance is opposed to the force, its tendency is to produce a constant increase of motion. If there is an opposing resistance, work is done in overcoming it, and if the force is greater than the resistance, the excess goes toward increasing the amount of motion. Work implies a change of position and is independent of the time taken to do it.

Energy is the power of doing work. If a steam engine can do more work than a man it is because it has more energy; for the same reason a cannon ball can overcome a greater resistance than a base ball because it has more energy. The unit of work is the foot-pound. By means of it work is represented numerically. Thus two elements must be considered when designating the amount of work; force and space. The foot-pound is the amount of work required to raise 1 pound to the height of 1 foot against the force of gravity. That is, the foot-pound is equal to about 32.16 foot-poundals, because the weight of a pound mass, or, the attraction of the earth for it is equal to this amount. If a weight of 10 pounds is lifted 1 foot from the ground, 10 foot-pounds of work have been performed, or if a weight of 1 pound is lifted 10 feet, the same amount of work has been done as before. Again the raising of 2 pounds through a height of 5 feet, is equivalent to 10 foot-pounds of work; and, if 20 pounds are lifted 50 feet, 1000 foot-pounds of work have been done, and so on.

The rate of doing work is called power. A horse-power represents the ability to perform 550 foot-pounds of work in a second, or

33,000 in a minute. An engine that could do 99,000 foot-pounds of work in one minute would be a 3 horse-power engine, because $\frac{99000}{33000} = 3$. A 3 horse-power engine does 1650 foot-pounds of work per second, because $\frac{1650}{550} = 3$.

Any moving body can overcome resistance, that is to say, it can perform work, because it has energy. A hammer striking upon a nail has velocity and is thus enabled to drive the nail, the amount which the nail is driven depending upon the weight of the hammer and upon its velocity. The energy of the hammer is expended in several ways, most of it in driving the nail into the wood; some of it, however, appears in the form of heat which raises the nail to a higher temperature and, to a certain extent, the hammer head as well. In the same way the energy of a cannon ball enables it to pierce, or overcome the resistance of an iron wall of considerable thickness. Whatever the amount of work done, or the manner of its performance does not in any way change the amount of energy expended. When energy is given out, work is done. We will now proceed to determine how much work is given out with various weights and velocities.

It has been seen that the amount of work done in raising a body weighing one pound to a height of 10 feet is 10 foot-pounds. When such a body falls it gives out, on striking the earth or point from which it started, 10 foot-pounds of energy, or if it falls from a height of 50 feet, 50 foot-pounds are given out when it strikes, and so on.

We may, therefore, find to what vertical height the velocity of a body would lift it and thus determine its energy, which is ordinarily given in foot-pounds. Suppose the velocity of a body to be denoted by v. If the height of fall necessary to give it that velocity is determined, then multiplied by the weight, the amount of energy given out when it strikes is obtained. As the velocity of a body is equal to 32.16 times the number of seconds that it has been falling, let $g = 32.16$; $t =$ the number of seconds of fall, and S space passed through. Then as

$$v = gt$$

$$t = \frac{v}{g}$$

$$\text{Also } S = \tfrac{1}{2} gt^2$$

Substituting the value of t just obtained we have,

$$S = \tfrac{1}{2} g \frac{v^2}{g^2} = \frac{v^2}{2g}$$

Since the energy is equal to the weight multiplied by the height, if we represent the weight by W and the height by S, the following very important formula is obtained:

$$\text{Kinetic energy} = WS = W\,\frac{v^2}{2g} = \frac{Wv^2}{2g} = \frac{Wv^2}{64.32}$$

The weight of a body is due to its mass and the force of gravity, hence, $W = mg$, in which m represents the mass. If we substitute m for $\frac{W}{g}$ in our formula for kinetic energy, we have kinetic energy $= \frac{1}{2} mv^2$.

Kinetic Energy is the energy of motion. There is also a *potential energy* or an *energy of position.* For instance, if a pound weight falls 50 feet, 50 foot-pounds of work are given out. If it is raised in the air and suspended 50 feet from the ground, it has *potential* energy, that is, the energy to accomplish, if set free, 50 foot-pounds of work.

Kinetic may be converted to potential energy. If a ball is thrown upward with a velocity of 64.32 feet it has a certain amount of kinetic energy. At the end of a second it has a velocity of only 32.16 feet, *i. e.*, its kinetic energy has diminished, but it has risen 48.24 feet, and has already a considerable potential energy. This potential energy is due to the kinetic energy which has disappeared. At the end of another second the ball has lost all of its velocity, and has reached the turning point where it is at rest. It has, therefore, no kinetic energy as it has no velocity, but the energy with which it started has not been lost. The ball has risen 64.32 feet into the air, and therefore has a potential energy. If the ball is held in that position the energy is kept in store as potential energy, and when it is allowed to fall, it gains in velocity and kinetic energy at the same time. When it reaches the ground at the end of 2 seconds, it has no potential energy but just the same amount of kinetic energy that it had when it began to rise. It is thus seen that the transformation or convertibility of energy is accomplished without any change in the total amount.

In the above example the sum of the kinetic and potential energy is the same at all times. At first it was all kinetic, then partly kinetic and partly potential, then all potential, then partly potential and partly kinetic, and finally all kinetic again, but the sum of the two was a constant quantity.

The energy of the world is constant. More or less of it may exist as kinetic or potential, but the sum of the two is always the same. This is illustrated very well by a pendulum. When it hangs vertically it has no energy at all; that is, considered as a pendulum. If we push it to one side it is raised a certain amount, and work or kinetic energy is expended upon it. That energy is stored as potential energy ready to be re-converted into kinetic whenever it is released. When it is allowed to swing, the energy is reconverted from potential to kinetic, and when it is at its lowest point, the energy is all kinetic again. This kinetic energy is expended in swinging it past its original position, and raising it through a certain distance. When that kinetic energy has been expended it stops, all the kinetic having been converted into potential energy again. In turn, this potential is re-converted to kinetic, which is again converted to potential, and so on. So the pendulum swings between the extremes of all potential and all kinetic but at every instant, and at every point of the arc traversed the total energy, or the sum of kinetic and potential is the same, and is always equal to the energy originally exerted in lifting it from its original position in the first place. From this it will be seen that energy is indestructible, and that, were it not for the friction and resistance of the air, the pendulum would vibrate forever. Energy is withdrawn from the pendulum to overcome these resistances. It is not destroyed, however, but is transformed into heat and other forms of molecular energy. Hence, the conclusion that energy is as indestructible as matter is correct.

A body weighing 100 pounds moves with a velocity of 20 miles per hour. Find its kinetic energy. Solution.—First find the velocity in feet per second. In this case it is $29\frac{1}{3}$. The kinetic energy of a weight is expressed by the formula,

$$K = \frac{W\ v^2}{64.32}$$

In the above example W is 100 pounds and v is $29\frac{1}{3}$, hence the formula becomes by substitution,

$$K = \frac{100\ (29\frac{1}{3})^2}{64.32}$$

solving K = 1337.7 + foot pounds.

Find the horse-power (H. P.) of an engine which does 14,300 foot pounds of work per second. Solution.—One H. P. = 550 foot pounds per second. Therefore to obtain the H. P. in the problem divide 14,300 by 550, giving 26 H. P. as the quotient.

EXAMPLES FOR PRACTICE.

1. What is the H. P. of an engine that can raise 1500 pounds 2376 feet in three minutes? Ans. 86 H. P.

2. What is the kinetic energy of a ball weighing 192.96 pounds, if it is rolled with a velocity of 100 feet per second? Ans. 30,000 foot pounds.

3. How long will it take a two horse-power engine to raise 5 tons 100 feet? Ans. 15.15 + minutes.

4. How far can a two horse-power engine raise 5 tons in 30 seconds. Ans. 3.3 feet.

CENTRIFUGAL FORCE.

If a weight is fastened to a string and whirled around, there is a pull on the string which is greater or less according as the velocity increases or decreases. In what follows the force of gravity will be neglected.

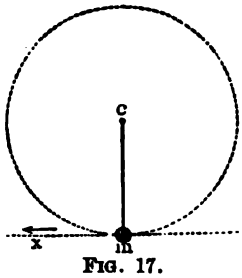


FIG. 17.

According to the first law of motion, a body tends to move in a straight line unless acted upon by some external force which causes a change in direction. In this case, the body would move off in a straight line on a tangent to the circle, as shown in the figure, unless it were prevented by the string which compels it to move in a circle.

According to the third law of motion, every force has an equal and opposite reaction. The force exerted by a body that is compelled to move in a curve is called the *centrifugal force*, since it tends to act away from the center of motion. The reaction or the tension of the string which prevents the weight from flying

away from the center, is called the *centripetal* force, as it acts toward the center of motion. The two forces of course tend to pull the string apart, and if the centrifugal force is great enough it will be broken. Nothing can revolve without the generation of centrifugal force. An illustration of this is to be found in the case of fly-wheels in mills, which have broken under the influence of centrifugal force.

Centrifugal force increases very rapidly with the velocity, and is directly proportional to the square of the same. It also increases with the weight of the body, and is directly proportional to the mass and inversely proportional to the radius. This relation may be expressed as follows: —

$$\text{Centrifugal force} = \frac{mv^2}{r}$$

r being the radius of the path.

To express this in gravity units we substitute for m its value $\frac{w}{g}$ thus obtaining the formula

$$\text{Centrifugal force} = \frac{wv^2}{gr}.$$

The velocity equals the number of revolutions times the circumference, divided by the time, the circumference being 3.1416 times the diameter. Hence the velocity may be obtained by the following formula

$v = \frac{3.1416DN,}{t}$ where D equals the diameter, N the number of revolutions, and t the time.

Find the centrifugal force of a 15-pound weight attached to a wire 3 feet in length, and revolving at the rate of 80 feet per second. Using the formula

$$\text{Centrifugal force} = \frac{wv^2}{gr}$$

$$\text{our equation becomes} = \frac{15 \times (80)^2}{32.16 \times 3}$$

$$= 995.+ \text{ pounds.}$$

EXAMPLES FOR PRACTICE.

1. What is the centrifugal force exerted by a weight of 40 pounds which moves in a circle whose diameter is 6 feet, and with a velocity of 24 feet per second? Ans. 238.8+pounds.

2. What is the centrifugal force of a 30-pound weight which makes 180 revolutions per minute, and is suspended 10 feet from an axis?

Ans. 3314.+ pounds.

Another Method. To find the centrifugal force of a given weight, multiply the square of the revolutions per minute by the radius of the circle, in feet, in which the weight revolves, and this product by the weight itself. This last product multiplied by the constant, .000341, will give the centrifugal force in terms of the weight of the body.

Example. A weight of forty pounds revolving 75 times per minute is suspended 3 feet from center of shaft. What is its centrifugal force?

Ans. $75^2 \times 3 \times 40 \times .000341 = 230.175$ pounds.

PRINCIPLES OF MACHINERY.

A Machine is a device by means of which force can be advantageously applied to a resistance. Its general use is to effect a transformation in the intensity of forces, so that one of small intensity acting through a long distance may reappear as a force of greater intensity acting through a shorter distance or *vice versa*.

For example, a force of 10 pounds acting through a distance of 50 feet may be transformed into a force of 20 pounds acting through a distance of 25 feet. The machine does not create or increase the energy, and more can never be taken out than is put into it, but, on the other hand, it actually loses some of that which it receives. This loss will be discussed under the head of efficiencies of machines. The loss is due to the waste of power necessary to overcome the resistance due to friction, air, raising temperature, etc. It must be remembered that the machine does not perform work of itself, and is not a source of power.

Many advantages are obtained from the use of machines. They permit of an exchange of intensity for a velocity which could not otherwise be obtained, and *vice versa*. An example of the latter is shown in the case of pulleys. They enable us to change the direction of our force, for, in order to *raise* a weight, we pull *down* on a rope.

Machines are also used to employ other forces than our own, such as the strength of animals, forces of wind and water, and to utilize the potential energy of coal by transforming it into kinetic energy through the medium of steam.

There are some principles which must always be remembered.

1. What is gained in intensity of power is lost in distance, time, or velocity, and what is gained in distance, time, or velocity, is lost in intensity of power.

2. The power multiplied by the distance through which it moves equals the weight multiplied by the distance through which it moves.

NOTE. The force applied is considered to be the power, and the work done the weight or load.

3. Power multiplied by its velocity equals the weight multiplied by its velocity.

LEVERS.

A Lever is an inflexible bar which can be freely moved about a fixed point (or line) called the fulcrum. The parts of the bar on each side of the fulcrum are called the arms of the lever.

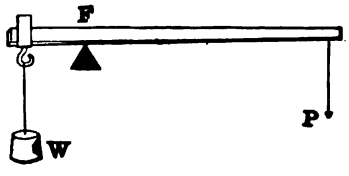


FIG. 18.

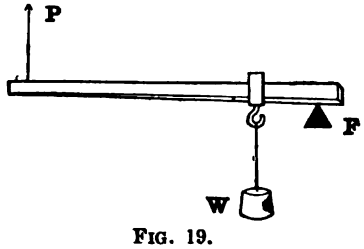


FIG. 19.

Each lever has two arms. In figure 18 the portions to the right and left of the fulcrum F are the arms of the lever. The power arm is the perpendicular distance from the fulcrum to the line in which the power acts. The weight arm is the perpendicular distance from the fulcrum to the line in which the weight acts. If the arms are not in the same straight line the lever is called a bent lever. There are three classes of levers as follows: —

In the first class, (Fig. 18), the fulcrum is between the power and the weight, and the power and the weight act in the same direction.

In the second case (Fig. 19) the weight is between the fulcrum and the power, and the weight and the power act in opposite directions.

In the third case (Fig. 20) the power is between the fulcrum and the weight, and the weight and power act in opposite directions.

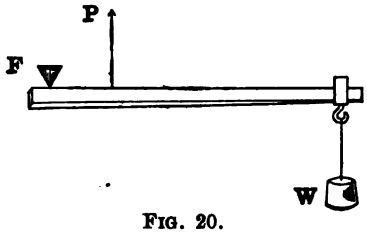


FIG. 20.

Before considering the laws of the lever, take a simple case. Suppose a plank to rest upon two supports, one at each end, and a weight is placed upon it. If the weight is half way between the supports, the pressure upon each support is the same. If it is nearer one support than the other, the weight is greater upon that support. It will be found that the weight bearing upon the supports is inversely proportional to the distance of the weight from these supports. In the case of the levers, the power and weight are inversely proportional to their corresponding arms. If two weights are hung upon a lever which is balanced from a fulcrum, it will be found that the product of the first weight and its lever arm is the same as the product of the second weight and its lever arm.

A given power will support a weight as many times greater or less than itself, as its arm is times as long as the arm from which the opposing weight is suspended. Hence, in a lever the power multipled by the power arm is equal to the weight multiplied by the weight arm.

NOTE. When a machine is in equilibrium so that there is no motion, as in the above cases, the laws which we deduce are termed static laws. If one of the products is greater, motion results, and the motion is governed by the law of kinetics, and this occurs, if, for instance, the power multiplied by the power arm is greater than the weight multiplied by the weight arm or vice versa; the difference of products is called the **resulting moment.**

The lever itself should be in equilibrium before the weight and power are applied. When the power is said to be multiplied by power arm it is not to be supposed that pounds are multiplied by feet, but the *number* of pounds by the *number* of feet.

It is not necessary that a lever should be straight. The power multiplied by its perpendicular distance from the fulcrum always equals the weight times its perpendicular distance from the fulcrum.

The Moment of a Force acting about a given point, (such as the fulcrum of a lever) is the product of the magnitude of the force and the perpendicular distance between the given point and the line in which the force acts. If, in the case of a lever, the power arm is 4, the weight arm 10, and the weight is 6, the moment of the weight about the fulcrum is 6×10 or 60, and as the moment of the power must be the same, the power must be $\frac{60}{4}$ or 15.

A lever is 10 feet long, and at one end there is a weight of 50 pounds. How much can be placed at the other end if the fulcrum is 6 feet from the weight, and yet maintain equilibrium?

Solution,—

$$6 \times 50 = 4 \times x.$$
$$4x = 300.$$
$$x = 75 \text{ pounds.} \quad \text{Ans.}$$

What is the resulting moment of a lever 24 feet long which has a weight of 30 pounds 6 feet from the fulcrum at one end, and force acting downward of 15 pounds at the other?

Solution,—

$$30 \times 6 = 180. \qquad 24 - 6 = 18.$$
$$18 \times 15 = 270.$$
$$270 - 180 = 90. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

1. Length of lever = 16 feet, weight at one end, 100 pounds; what power applied at the other end, $3\frac{1}{2}$ feet from the fulcrum, is required to move the weight? Ans. 357.14 + pounds.

2. The power arm is 18 inches; the weight arm is 21, and the weight is 24 ounces. What is the power? Ans. 28 ounces.

3. The weight arm is 1 foot, the power is 50 pounds, and the weight is 2,500 pounds. What is the power arm? Ans. 50 feet.

4. The resulting moment of a lever of the first class is 120. The weight arm is 6, the weight 30, and the power arm is 20. What is the power? Ans. 15 or 3.

5. What is the moment of a bar 160 feet long, attached at one end to a beam, and with a weight of 4 pounds at the other? Ans. 640.

PULLEYS.

A Pulley consists of a wheel turning upon an axis and having a cord passing over its circumference. The frame which supports the axis of the wheel is called the block. The advantages of a pulley depend upon the uniform tension which exists throughout the cord. If a cord is passed over a fixed pulley as in Fig. 21, and a weight is hung from one end while the other is held fast, the pull on the end which is held is equal to the weight which is suspended at the other end. If the weight is moved by the power, it and the weight traverse equal distances. It is evident, then, that the fixed pulley does not afford an increase in power, but simply a change in direction. A movable pulley is illustrated in Figs. 22 and 23. The weight hangs from the movable pulley, and one end of the cord which supports the pulley is fastened so that it cannot move, while the other end passes to the hand or power. One-half the weight is therefore held by the fixed end of the cord, and one-half by the hand or power, hence the tension in the cord is only one-half that of the first case. It is also evident that, in order to raise the weight a given distance, the power must traverse twice that distance, because the hand travelling through a certain distance has to shorten twice the length of cord that it would have to shorten in the first case, that is, in order to raise the weight 1 foot, the hand must traverse 2 feet. This shortens each section of the cord 1 foot, and thus raises the weight 1 foot. By this arrangement it will be seen that the velocity of the power is greater, but the force required is less than in the first case. That the force is less is also obvious from the fact that the end of the cord rigidly fastened supports one-half of the weight. Thus, there is no increase of energy from the lesser force neces-

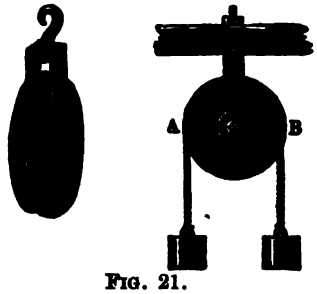


FIG. 21.

sary, because what is gained in power is lost in velocity. In the case of Fig. 24, the power is proportioned to the weight as one is to two raised to a power corresponding to the number of pulleys. Thus, 1 pound at p will balance 8 pounds at q for there are three movable pulleys and $2^3 = 2 \times 2 \times 2 = 8$.

If there are a number of pulleys with a single cord as in Fig. 25, the tension of the cord is uniform, and each section of it assists in supporting the weight. Hence, the greater the number of sections of cord, the less the tension in the same, but the greater the distance which the power must move to raise the weight a given distance. That is, the greater the number of pulleys, the greater the weight which can be supported by a given force.

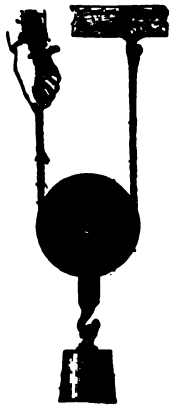


FIG. 22.

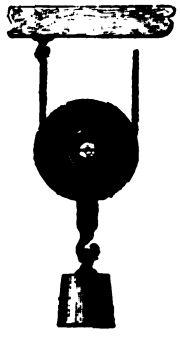


FIG. 23.

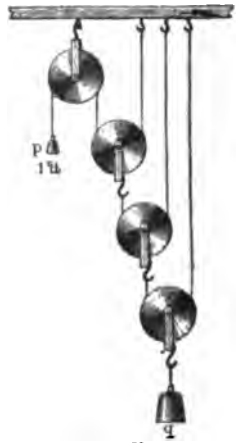


FIG. 24.

Ignoring friction, in a pulley having a continuous cord, a given power will support a weight as many times as great as itself, as there are parts of the cord supporting the movable block. In Fig. 25 it is evident that, when the fixed end of the cord is attached to the fixed block, the number of parts of the cord supporting the weight is twice the number of movable pulleys used, and that when the same end of the cord is attached to the movable block the number of parts of the cord is one more than twice the number of movable pulleys used.

To determine what force will support a given weight; *Divide the weight by the number of parts supporting the movable pulley.*

To determine what weight can be supported by a given power; *Multiply the power by the number of cords supporting the movable block.*

If the power is greater than that necessary to support the weight, the weight is raised. If the weight is greater than the power necessary to hold it, it descends. In each case the velocity increases if both power and weight remain the same, because the force which is in excess of the amount necessary to preserve the state of rest or motion, always acts to give an increased velocity.

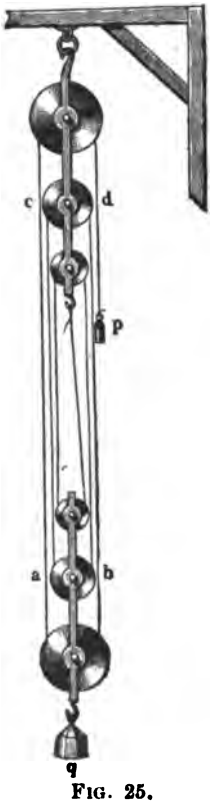


FIG. 25.

Given a 3-cord pulley and a power of 250 pounds. What weight will it support?

Solution,—

$250 \times 3 = 750$ pounds. Ans.

EXAMPLES FOR PRACTICE.

1. Given a power of 90 pounds, what weight can be supported by a 4-cord pulley?

Ans. 360 pounds.

2. The pulley has 7 cords, and we wish to raise a weight of 3 tons. What force is necessary?

Ans. $857\frac{1}{7}$ pounds.

3. The force available is 96 pounds; the pulley has 3 cords. What weight can be lifted?

Ans. 288 pounds.

WHEEL AND AXLE.

The Wheel and Axle consists of a wheel and a cylinder united in such a way that they revolve about a common axis. It is really a modified lever as will be seen. The ordinary range of action of a lever is very small. Theoretically we see that the weight could not be raised more than twice the length of the weight arm, but usually in practice, it would not be possible to obtain so great a range.

The wheel and axle, however, has the advantage of being able to raise the weight through a greater distance. In Fig. 26, it is seen that the rope acts with a certain pull p through the power arm cd, which is half the diameter of the wheel, while the weight acts through the much smaller weight arm ab which is half the diameter, or radius of the axle. If the product of the force p and the radius of the wheel cd exceeds the product of the weight r and the radius of the axle ab, the weight is raised. If the diameter of the wheel is 5 times the diameter of the axle a pull of 100 pounds will hold in suspension 500 pounds, because the power arm, or radius of the wheel, is 5 times the weight arm, or radius of the axle.

It is seen that the force p would have to act through a distance equal to the circumference of the wheel, while the weight acts only through a distance equal to the circumference of the drum for each revolution of the wheel and axle. The force times the distance through which it moves equals the weight times the distance through which it moves. These distances are proportional to the radii of the wheel and axle, which may be expressed as, $p \times cd = W \times ab$. The weight and the force necessary to support it are inversely proportional to their respective arms, *i. e.*, to the radii of the wheel and axle. Written as a proportion it becomes $p : W : : ab : cd$.

FIG. 26.

If instead of a wheel at the end of the axle we have cranks, the apparatus is called a windlass. In this case the lever arm of the force is the portion of the arm or crank from the point where the force is applied to the center or axis of the axle, while the weight, as before, acts with the radius of the axle as the weight arm. A capstan differs from the windlass only in that it exerts its pull from a vertical position. It is used for hauling heavy weights or hoisting anchors aboard ships.

The preceding may be summarized as follows: —

To calculate the lifting power of a wheel and axle; ***Multiply***

the force by the radius of the wheel and divide the result by the radius of the axle.

To calculate the force required to hold a given weight in equilibrium with a wheel and axle; *Multiply the weight by the radius of the axle, and divide by the radius of the wheel.*

Let R equal the radius of the wheel, r equal the radius of the axle or drum, W the weight to be raised, and F the force applied to the wheel. Then $W : F :: R : r$

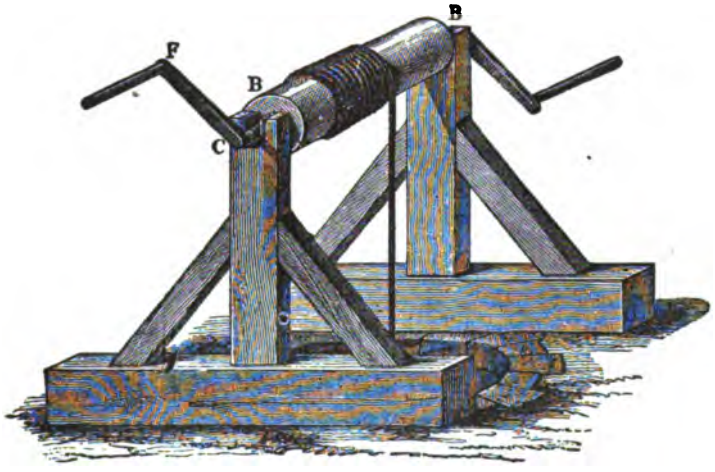


FIG. 27.

What weight will a force of 20 pounds raise when the radius of the axle is 3 inches, and that of the wheel 30 inches.

Solution,—

$$30 \times 20 = 600$$
$$600 \div 3 = 200 \text{ pounds.} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

1. How much force will be required to lift a weight of 230 pounds if the axle diameter is 10 inches, and that of the wheel 70 inches. Ans. $32\frac{6}{7}$ pounds.

2. The wheel of a windlass is of 28 inches radius. A force of 100 pounds will lift a weight of 1,400 pounds What is the radius of the axle? Ans. 2 inches.

THE INCLINED PLANE.

The Inclined Plane is a smooth, hard, inflexible surface inclined so as to make an oblique angle with the direction of the force to be overcome. In most cases it is a plane surface making an acute angle with the horizontal, and is used in overcoming the force of gravity. When a weight rests upon an inclined plane, the force of gravity tends to draw it vertically downward. This force is resisted partly by the plane, *i. e.*, by a force perpendicular to the inclined plane. The other component, or the remainder of the force of gravity, tends to draw the object resting on the plane downward over its surface. If the body is free to move, movement down the plane will result. This resolution of the force of gravity is shown by Fig. 28.

AB represents the force of gravity acting on the ball which rests on the inclined plane; AC, the force with which the ball presses against the inclined plane. This is of course resisted by the reaction of the plane which is equal to it in amount. AD is the other component of gravity which, if not opposed by some other force, will cause the ball to roll down the plane, in the direction in which it acts. There are three cases of the use of the inclined plane.

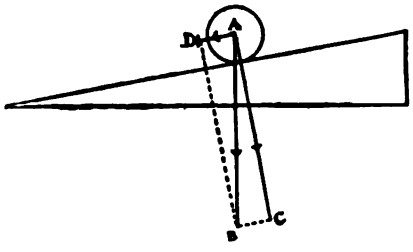


FIG. 28.

1. Where the power acts in a direction parallel to the length of the plane.

2. Where the power acts in a direction parallel to the base of the plane (generally horizontal).

3. Where the power acts in a direction parallel to neither the length nor the base of the plane.

The first case is illustrated in Fig. 28 where the line AD represents the force tending to cause the ball to roll down the plane. A force equal to this but in the opposite direction is necessary to hold it in equilibrium, and is easily ascertained. The line AD is in the same proportion to AC as the height (h) of the plane is to the base (b). AB is in the same proportion to these

two lines that the hypothenuse l of the triangle is to the height and base, giving the following proportion:

$$AD : AB : : h : l$$

If the force AD is represented by P, and the force AB (the weight of the ball) by W, we have

$$P : W : : h : l \text{ or } P = \frac{h}{l} W$$

If the height of the plane were one-half of its length or the hypothenuse of the triangle, the weight necessary to hold the ball in equilibrium would be equal to one-half of its weight. This is illustrated in Fig. 29. Here the line which prevents the ball from rolling down has a tension of 1 pound (half the weight of the ball) from the action of the weight of 2 pounds acting over the pulley. The following law may now be formulated for the first case:

When a given power acts parallel to the plane, it will support a weight as many times as great as itself as the length of the plane is times as great as its vertical height.

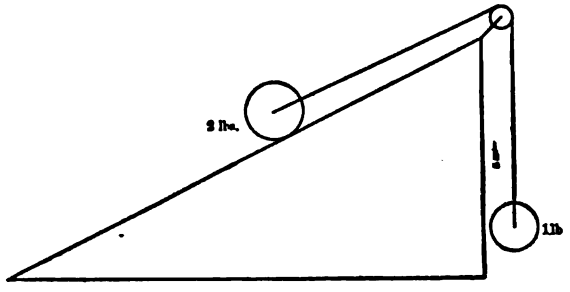


FIG. 29.

This may be seen more readily by an observation of the fact that in order to allow the ball which rests on the plane to descend from the top to the bottom of the plane, the other force must rise twice the height of the plane if the length of the plane is twice the height. Suppose the plane to be 2 feet in length and 1 foot in height, and the weights of the ball resting on the plane and the balancing force 20 and 10 pounds respectively. The 20 pound ball descends a distance of 1 foot, although it traverses 2 feet in order to accomplish this. Thus the work done is 20 foot-pounds. In order to permit the 20 pound ball to descend that

distance the weight of 10 pounds rises a distance of 2 feet. Here again the work done is 10 × 2 or 20 foot-pounds.

In the second case, the power is supposed to act parallel to the base. *A given power acting parallel to the horizontal plane, will support a weight as many times greater than itself as the horizontal base of the plane is times as great as its vertical height.* This may be seen from Fig. 30, in which the force AB is resolved into the forces AC and AD. The force AE represents the force required to maintain equilibrium. It is greater than the force AF (the force acting to cause the weight to move down the plane)

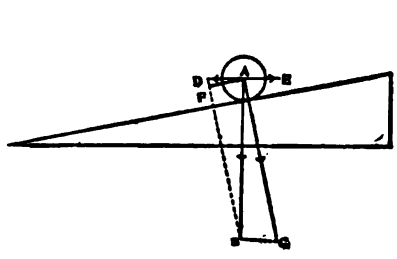


FIG. 30.

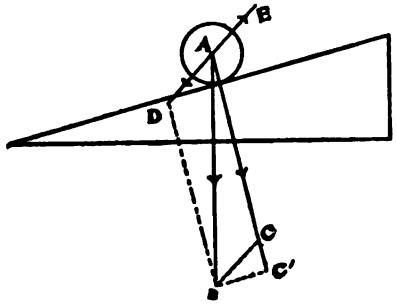


FIG. 31.

because a part of the balancing force AE is expended in increasing the pressure of the weight against the plane, *i. e.*, increasing the force AC. This would also be true of any force tending to press the weight towards the plane, which is always the case unless it acts parallel to the plane, or in a direction away from it.

The third case can only be determined by trigonometry or graphically by drawing a parallelogram of forces, as shown in Fig. 31. The power in this case must be greater than if it acts parallel to the plane, because it has the effect of decreasing the force with which the weight presses against the same. This is evident from the illustration, because if the force acted parallel to the plane, the line BC would lie in the position indicated by the dotted line BC′ and AC would become equal to AC′. This may be more readily comprehended by comparison with Fig. 28 in the first case.

THE WEDGE.

A Wedge is a movable inclined plane in which the power usually acts parallel to the base. It is used for moving great weights short distances, and the law is the same as for the corresponding inclined plane. A common method for moving bodies is to place two similar wedges as in Fig. 33 with their thin ends overlapping under the load, simultaneous blows of approximately equal force being struck upon their heads. In this case the same force must be used upon each wedge as though only one were used. But since the power is double and as the weight remains the same, the distance the body is moved is twice as great as if there were but one wedge. Another common use of the wedge is for splitting timber, as is illustrated in Fig. 32. The longer the wedge for a given thickness the greater the gain in intensity of power.

FIG. 32.

Let ABC, Fig. 34, represent a wedge, F the force against which it acts, and R the reaction equal to the force F. Suppose now that a blow is struck against BC so that the wedge is driven in a distance equal to the distance between the two dotted lines. The

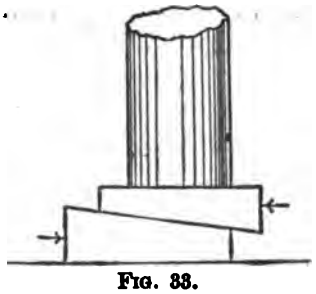


FIG. 33.

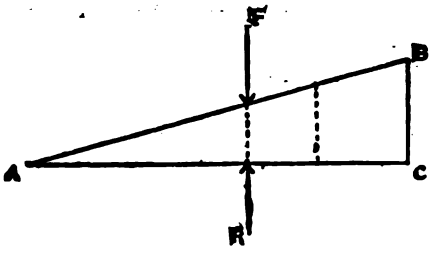


FIG. 34.

force F is then opposed through a distance equal to the difference in height of the dotted lines. If the distance between the dotted lines is 1 inch and the difference in the height of the dotted lines $\frac{1}{8}$ of an inch, the blow required to push the wedge in must be slightly greater than $\frac{1}{8}$ of the force F, that is, the force F is not

quite 8 times as great, because a certain amount of force is necessary to produce motion. Heretofore friction has been disregarded, but this makes a difference in the amount of force necessary to drive the wedge, and will be considered later.

The height of an inclined plane is 6 inches, the length is 3 feet, what force parallel to the length is necessary to prevent a weight of 15 pounds from moving down the plane?

Solution—

$$h = \tfrac{1}{2} \text{ foot} \quad l = 3 \text{ feet}$$
$$W = 15 \text{ pounds}$$
$$P = \frac{15 \times \frac{1}{2}}{3}$$
$$= 2\tfrac{1}{2} \text{ pounds.} \quad \text{Ans.}$$

The height is 16 inches, the base is 56 inches, and the power is 10 pounds parallel to the base. What weight will this power support?

Solution—

$$10 = \frac{16\ x}{56} \quad \text{where } x = \text{weight}$$
$$560 = 16\ x$$
$$x = 35 \text{ pounds.} \quad \text{Ans.}$$

A wedge whose central length is 12 inches and whose height is 2 inches, has a force of 35 pounds behind it. What weight can be raised?

Solution—

$$35 = \frac{2\ x}{12} \qquad 2\ x = 420$$
$$x = 210 \text{ pounds.} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

1. The force acting parallel to the base of an inclined plane is 50 pounds, the height is 6 feet and the base is 27 feet. What weight will the force hold in equilibrium? Ans. 225 pounds.

2. In the above example the weight is 50 pounds. What force is necessary to prevent motion? Ans. $11\frac{1}{9}$ pounds.

3. An inclined plane is 17 feet in length. Its height is 9 inches. What weight will balance a force of 34 pounds acting parallel to the length? Ans. $770\frac{2}{3}$ pounds.

4. If the length and height are the same as in the previous example, what force will hold a weight of 170 pounds at rest?

Ans. $7\frac{1}{2}$ pounds.

5. A wedge is struck a blow of 42 pounds. Its base is 16 inches, and its greatest thickness is 3 inches. What weight must be put upon its side to prevent motion disregarding friction?

Ans. 224 pounds.

SCREWS.

A screw is a cylinder with a helical groove running around its circumference, as in Fig. 35. The letters show the positions taken by the points *a*, *b*, *c*, *d*, *e*, and *f*, when the line *a f* passes around the cylinder. This helix is termed the thread of the

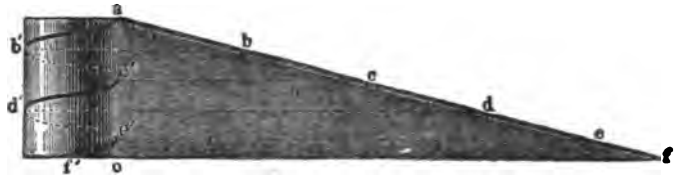


FIG. 35.

screw. The distance that a point on this helix is advanced or withdrawn in one complete turn of the screw is called the pitch. Figures 36 and 37 are examples of the screw.

Fig. 39 represents a screw-jack. The screw is turned in the nut A, the force to turn it being applied at the point on the

FIG. 36.

FIG. 37.

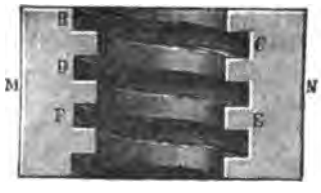


FIG. 38.

handle marked P. For one complete revolution of the handle a point on the helix is advanced in the direction of the axis of the screw an amount equal to the pitch, so that the weight W is raised by that amount. During the time that the weight is raised a distance equal to the pitch, the point P passes through the circumference of a circle whose radius is R. In every case, the

force applied multiplied by the distance through which it travels, is equal to the weight moved multiplied by the distance which it traverses, W times the pitch of the screw equals P times the circumference whose radius is R. With the screw a given power supports a weight as many times greater than itself as the circumference described by the power is times as great as the pitch of the screw. This may be expressed as follows:

$$\text{Force P} = \frac{Wp}{C}$$

where W is the weight, p the pitch of the screw and C the distance through which the power moves.

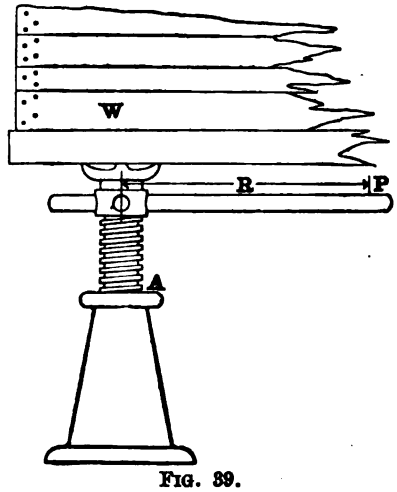


FIG. 39.

Suppose we have a screw with a pitch of $\frac{1}{4}$ inch, a power of 18 pounds applied to an arm 3 feet in length. How much weight can be lifted?

Solution—

$$18 = \frac{\frac{x}{4}}{36 \times 2 \times 3.1416}$$

$$x = 16286.0544 \text{ pounds.}$$

EXAMPLES FOR PRACTICE.

1. The pitch of a screw-jack is $\frac{1}{2}$ inch, the power of 10 pounds goes through a circumference of 20 feet in one revolution. How much weight can be lifted? Ans. 4800 pounds.

2. Given a weight of 6400 pounds, a screw with a pitch of $\frac{3}{8}$ inch, a power arm of 2 feet length. What power is necessary to raise the weight? Ans. 16—pounds.

Single threaded screws of less than an inch pitch are usually classified by the number of threads in one inch of length. To obtain the pitch in such a case, the inch is divided by the number of threads. The pitch of a screw with 5 threads per inch is $\frac{1}{5}$ of an inch; with 10 threads, $\frac{1}{10}$ of an inch, and so on.

FRICTION.

Friction has, heretofore, been neglected, though it is the principal obstruction to the motion of a machine. It is the resistance which a moving body encounters from the surface on which it moves. An absolutely smooth surface cannot be produced by any known means. Even a polished surface contains minute projections which fit into corresponding depressions if it rests on another surface, and to produce motion the resistance produced by these projections and depressions must be overcome, *i. e.*, the projections must be lifted out, bent down, or broken off. The following facts concerning friction have been determined by experiment.

1. Friction is greatest at the beginning of motion, because the projections of one surface have had an opportunity to sink more deeply into the depressions of the other.

2. Friction increases with the roughness of the surfaces.

3. Friction is greater between soft bodies than between hard ones.

4. Friction is nearly proportional to pressure, that is, the more heavily a brick, or similar substance, is pressed against a board, the greater will be its friction in pushing it along the board, for it is as easy to push two bricks side by side, as to push one brick with another one on top of it.

5. Friction is not affected by the extent of surface except within extreme limits, in other words the force necessary to push the brick would be the same whether it lay upon a side, on end, or on edge, unless the edge cut into the supporting surface.

6. Friction is greater between surfaces of the same material than between those of different materials, because the bodies have the same molecular structure. Hence, the little projections and depressions fit together more easily, like the teeth of similar saws. For this reason the axles of railway cars being made of steel, the boxes on which they revolve are made of brass, or some other metal, ordinarily an anti-friction one.

7. Rolling friction is less than sliding friction.

8. Friction is diminished by polishing or lubricating the adjoining surfaces. The joints of men and animals are lubricated by a liquid called the synovial fluid, which is a case where

friction is reduced to a minimum. The moving parts of engines where contact is made with other surfaces, are oiled. This is a common illustration of lubrication. When a journal becomes heated and expands, pressure is increased and thus friction is increased.

That friction is proportional to pressure is illustrated by many common incidents. The reason why an engine can move a train of empty cars more easily than if they were loaded is due to the fact that the load increases the resistance of friction.

Under a given pressure the friction is the same whether it is concentrated on a small area or distributed over a large one. For this reason the guides for engine crossheads, the bearings of cranks, etc., are made large, so that the pressure will be less per unit of area. The friction will be the same, but as it is distributed over a considerable area, there will be less wear and less heating.

As friction is less between surfaces of different materials, the rubbing parts of engines and machinery are usually made of dissimilar materials, brass bearings for steel shafts, for instance.

The starting of a train illustrates the fact that friction is greatest at the beginning of motion. The bearings of watches are made of jewels or hard steel, because friction is less between hard than between soft bodies.

Railroad trains, carriages, etc., are placed on wheels to reduce friction because friction is less when the bodies roll on each other than if one slides over the other.

If oil, grease, or graphite is placed on a surface, friction will be greatly reduced, the hollows being filled with some substance whose particles move on each other with little friction. The friction between liquids is much less than between solids and the friction between lubricated surfaces is much more a friction of the lubricant than a friction of the surfaces. For this reason bearings, guides, and the insides of cylinders are lubricated.

The ratio between the resistance due to friction and the normal or perpendicular pressure is called the *coefficient of friction.*

For instance, if a weight of 30 pounds rests on a horizontal plane and it takes 10 pounds to start the weight, the coefficient of friction between the weight and the surface is $\frac{10}{30}$ or .33.

Suppose the maximum pressure between a crosshead and the guide is 7,500 pounds and the coefficient of friction .2; then the amount of friction is $7{,}500 \times .2 = 1500$ pounds. Suppose the area of the slipper surface is 50 square inches, then the amount of friction per square inch will be 30 pounds. If the area is increased to 100 square inches, the friction per square inch will be 15 pounds or only one half as much as before, and the wear and liability to heat will, also, be only one half as great.

Friction is sometimes of great service. The brakes on electric cars and railroad trains, the belt over pulleys and the driving wheels of locomotives are instances of the utilization of frictional resistance.

The following table gives the coefficient of friction of various materials:*

Materials in Contact.	Coefficient of Friction.
Wrought iron on wrought iron	.25 to .41
Wrought iron on cast iron	.28 to .38
Steel on cast iron	.30 to .40
Brass on cast iron	.21 to .23
Timber on stone	.40
Iron on stone	.30 to .70
Timber on timber	.20 to .50
Timber on timber (soaped)	.04 to .20
Leather on metals (dry)	.56
Leather on metals (wet)	.36
Leather on metals (greasy)	.23
Leather on metals (oily)	.15
Cast iron on cast iron (lubricated)	.03 to .054
Cast iron on bronze (lubricated)	.03 to .054
Wrought iron on bronze (lubricated)	.03 to .054
Bronze on bronze (lubricated)	.09

If we wish to find the frictional resistance between two bodies, we multiply the weight or pressure of the bodies by the coefficient of friction between them.

*Rankine.

For instance, if a block of wood weighs 300 pounds and rests on a block of stone, we find the frictional resistance by multiplying 300 by the coefficient of friction between the two bodies or, from the above table, $300 \times .40 = 120$ pounds.

If we wish to find the amount of power expressed in foot pounds per minute, multiply the weight by the coefficient of friction as above and then multiply the result by the velocity in feet per minute.

Suppose the pressure on the box of a freight car is 9,600 pounds, and the coefficient of friction is .05. What will be the power absorbed when the journal speed is 210 feet per minute?

9,600 pounds multiplied by .05 is 480 pounds and the 480 pounds multiplied by 210 feet is 100,800 foot pounds.

If we want the horse-power absorbed, divide the number of foot pounds per minute by 33,000 or from the above problem

$$\frac{100{,}800}{33{,}000} = 3.054 \text{ horse-power.}$$

EXAMPLES FOR PRACTICE.

1. What pull will be required to haul a weight of 10,500 pounds over soaped timber, if the coefficient of friction is .15?

Ans. 1,575 pounds.

2. What is the frictional resistance offered by a bearing when the pressure on the pillow block is 10,000 pounds, the coefficient of friction being .06. Ans. 600 pounds.

3. What will be the foot pounds of resistance of the shaft if the pressure is 9,500 pounds, the coefficient of friction .05 and the speed 230 feet per minute? Ans. 109,250 foot pounds.

4. What horse-power will be required to turn a shaft $2\frac{1}{2}$ inches in diameter making 240 revolutions per minute, against frictional resistance, the pressure being 1,268 pounds and the coefficient of friction being .06? Ans. .36 + H. P.

TRANSMISSION OF MOTION.

There are several ways of transmitting circular motion from one shaft or axis to another. If the shafts are near together, it may be done by rolling contact or by gear wheels, sometimes called simply gears. If the distance requires wheels of large

diameter, rope or belt gearing is used. If the distances are great, electricity is the most economical means.

ROLLING CONTACT AND GEARS.

The simplest method of transmitting circular motion is by rolling contact, that is, if the edge of a revolving circular disc or wheel is pressed against the edge of another, the second will revolve also. The former, or the one to which the power is applied, is called the driver and the latter, or the one which receives the motion, is called the driven or follower. If the resistance of the

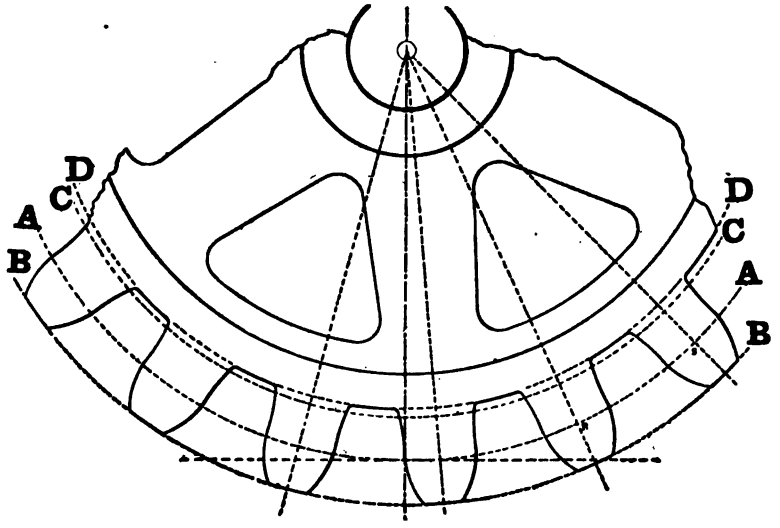


FIG. 40.

follower is considerable, the power of the driver will not move it, as the friction will be insufficient. For this reason, it is usual and often necessary to use wheels which are provided with teeth corresponding and meshing with each other. The teeth being of such a form that the motion will be very nearly the same as that of two circles rolling on each other. Assuming the motion to be the same as rolling contact, imaginary circles of contact, called pitch circles, are conceived to exist. Fig. 40 shows the outline of a portion of a gear; A A is the pitch circle, B B, the addendum circle, arcs of which form the tops or ends of the

teeth, and C C the root or dedendum line. In order to give a little clearance, to allow for inaccuracies of workmanship, the spaces between the teeth are cut down to the clearance circle D D. The length of teeth in the direction of the radius of the wheel, is divided into two parts; the portion between the pitch line and the top is called the addendum and the portion between the pitch line and the root circle is called the dedendum. The pitch is the distance from center to center of two teeth measured on the pitch circle R S, Fig. 40, or it is the distance from the face of one to the face on the other, measured on the pitch circle. Again, it is the width of a space and a tooth. The total depth of a tooth is about $\frac{2}{4}$ of the pitch. The addendum is .35, the dedendum .40 and the clearance .05 of the pitch. The space is wider than the tooth by a small amount, called the back-lash, which is usually $\frac{1}{8}$ to $\frac{1}{10}$ of the pitch. In finely-cut gears, or in those having a large pitch, the back lash is made very small.

In order that the teeth may run smoothly and with constant relative velocity, it is necessary that they should work with as little friction as possible; this result is obtained if the teeth roll over each other instead of sliding, which may be accomplished if the outlines of the teeth are formed of certain curves; when they will roll over each other with little friction. The curves in common use are the cycloid and involute. Teeth formed of involute curves are stronger at the root than those of the cycloid, and they may be spread apart, so that the pitch circles do not run tangent to one another, without seriously affecting their action.

If the teeth on a gear are parallel to the axis or shaft of the wheel, it is called a spur gear, shown in Fig. 40. When the motion is to be transmitted at an angle, bevel gears Fig. 41 are used; if the bevel gears are of the same diameter and have the same number of teeth, they are called mitre gears. The angle that the teeth of mitre gears make with the axis is 45 degrees, and they neither increase nor diminish relative velocity between the two shafts. Another method of transmitting power at right angles is by worm gear, shown in Fig. 42. As the worm is simply a screw, each revolution of the worm will rotate the worm wheel, or gear wheel, a distance equal to its pitch. For instance, if there are 50 teeth on the worm wheel, one revolution of the worm will

move the wheel $\frac{1}{50}$ of its circumference, or it will take 50 turns of the worm to turn the wheel once. This method is used when the velocity is to be considerably reduced.

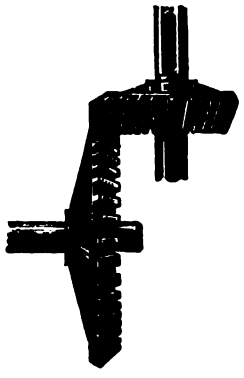


FIG. 41.

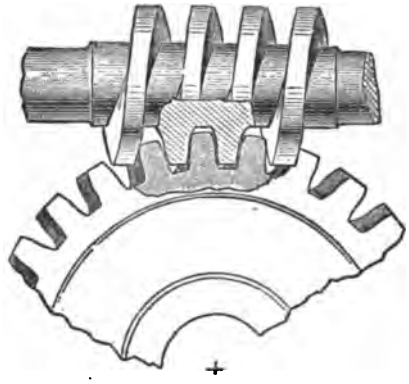


FIG. 42.

A small wheel which gears with a large one is called a *pinion*. Gears are often used to transmit motion, changing it from circular to linear. To do this, teeth are cut in a straight bar, so that a pinion will gear with it, and the combination is called a rack and pinion. The rack may be considered as the circumference of a circle of infinite diameter, and its teeth are occasionally made with straight instead of curved sides.

Helical gearing is a modification of ordinary gearing in which the pitch surfaces may be cones or cylinders, as in bevel and spur gearing. Fig. 43 shows double helical gear. This gear works very smoothly and quietly, as the teeth have always two points touching in the plane of the axes, besides being stronger than those of ordinary wheels.

When wheels are run at high velocity, the teeth of one wheel of each pair are generally made of wood and are called cogs. Such a wheel is called a mortice gear, and is used when noise is undesirable.

In the following discussion, let P be the pitch, N be the number of teeth and D be the pitch diameter, that is the diameter of the pitch circle.

If we know the pitch of a gear and the number of teeth, the pitch diameter can be found by multiplying the pitch by the number of teeth and dividing by π, or

$$D = \frac{P\ N}{3.1416}$$

For example: Suppose the pitch is 1.125 inches and there are fifty teeth. What is the diameter?

$$D = \frac{PN}{3.1416} = \frac{1.125 \times 50}{3.1416} = 17.9 \text{ inches.}$$

This formula may be used to find the number of teeth if we know the pitch and the diameter.

FIG. 43.

$$N = \frac{3.1416 \times D}{P}$$

How many teeth are there on a gear wheel when the diameter is 40 inches and the pitch is 1.675 inches.

$$N = \frac{3.1416 \times 40}{1.675} = 75 \text{ teeth.}$$

To find the pitch when the diameter and the number of teeth are known.

$$P = \frac{3.1416 \times D}{N}$$

Suppose the diameter of a wheel is 36 inches and there are 75 teeth. What is the pitch?

$$P = \frac{3.1416 \times 36}{75} = 1.508 \text{ inches.}$$

If we have two wheels that gear together, the product of the speed and the number of teeth of the driver equals the product of the speed and the number of teeth of the follower. Let R equal the number of revolutions per minute of driver, T equal number of teeth of driver, r number of revolutions per minute of follower, and t number of teeth of follower.

$$\text{Then } RT = rt.$$

If we have any three given to find the fourth, we simply transpose the quantities in the formula.

$$\text{To find R,} \qquad R = \frac{rt}{T}$$

Suppose the driver has thirty teeth and the follower has fifteen teeth and makes 50 revolutions per minute. How many revolutions does the driver make?

$$R = \frac{rt}{T} = \frac{50 \times 15}{30} = 25 \text{ revolutions.}$$

If the driver is a pinion with 11 teeth and makes 200 revolutions per minute; the follower has 55 teeth, how many revolutions will the follower make?

$$r = \frac{RT}{t} = \frac{200 \times 11}{55} = 40 \text{ revolutions.} \quad \text{Ans.}$$

To find the number of teeth of the driver when the speed of the driver, the number of teeth and the speed of the driven are known.

$$T = \frac{rt}{R}$$

Suppose the driver makes 40 revolutions and the follower has 32 teeth and makes 80 revolutions. How many teeth has the driver?

$$T = \frac{rt}{R} = \frac{80 \times 32}{40} = 64 \text{ teeth.} \quad \text{Ans.}$$

A driver makes 400 revolutions and has 12 teeth; the follower makes 96 revolutions. How many teeth has the follower?

$$t = \frac{RT}{r} = \frac{400 \times 12}{96} = 50 \text{ teeth.}$$

Trains of gear wheels are used for hoisting, for obtaining an increased or decreased velocity and for changing the direction of

motion. Each pair has its driver and follower. If the shafts are parallel, the usual arrangement for obtaining a change in velocity is to fasten two wheels of unequal size on every axis except the first and last, and for hoisting, to make the smaller wheel of every pair, gear with the next larger of the series. The power applied at the first small wheel or pinion, with high speed, will slowly raise the weight, if a rope or chain is wound about a drum or barrel fixed to the axis of the last large wheel. If the power is at low speed and it is desired to get high speed, the process is reversed.

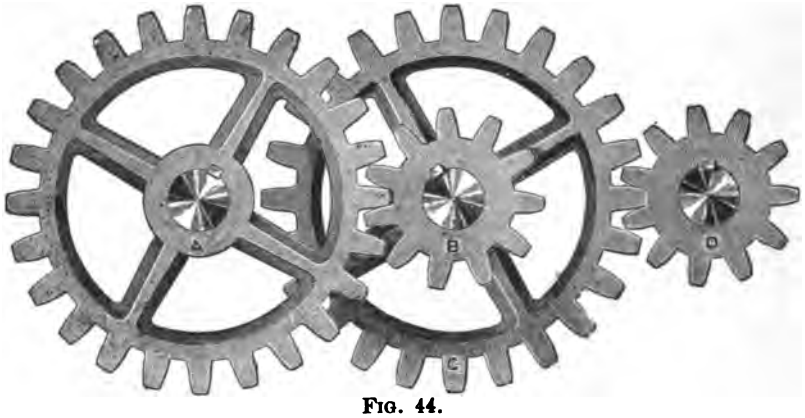


FIG. 44.

Figure 44 shows a train composed of four wheels, for an increase in velocity. A and C are drivers and B and D are followers. To find the number of revolutions that D makes while A makes one, we divide the product of the number of teeth in the drivers by the product of the number of teeth in the followers.

Suppose A has 30 teeth, B has 10 teeth, C has 22 teeth, and D has 11 teeth, then

$$\frac{30 \times 22}{10 \times 11} = 6$$

equals the number of revolutions that D makes while A makes one, or it is the velocity ratio of the last and first shafts.

Suppose the gears for an electric travelling crane are to be designed. The motor makes 1000 revolutions per minute and a weight of 10 tons is to be raised at the rate of 20 feet per minute, using 3 pairs of gears.

If it is assumed that the drum on which the chain is wound is $1\frac{1}{4}$ feet in diameter; then for every revolution, the weight will rise $1\frac{1}{4} \times 3.1416$ or about 4 feet; therefore, the drum must make $\frac{20}{4}$ or 5 revolutions per minute. The first shaft makes 1000 revolutions and the last makes 5; to reduce this velocity by gears is the problem. $\frac{1000}{5} = 200$, or the first shaft makes 200 revolutions while the last makes one revolution. Since the product of the number of teeth of all the followers divided by the product of the number of teeth of all the drivers will give the velocity ratio, then, if three pairs are adopted their velocity ratio must be 200. Let us assume the following: The pinion A with 15 teeth, making 1000 revolutions, drives B with 60 teeth. On the same shaft with B is keyed the pinion C, having 16 teeth, and D with 80 teeth gears with it. The pinion E with 14 teeth is keyed to this latter shaft, and drives a gear wheel F, having 140 teeth. The drum is fixed to the shaft on which is the wheel F; then, —

The product of the teeth of the followers, divided by the product of the teeth of drivers equals the velocity ratio, or 200.

$$\frac{B \times D \times F}{A \times C \times E} = \frac{60 \times 80 \times 140}{15 \times 16 \times 14} = 200$$

Thus with the above gears, the weight will be raised nearly 20 feet per minute.

HORSE POWER OF GEARS.

The strength of gear teeth and the horse-power that may be transmitted depends upon many things that may be variable and uncertain. The formulas deduced have a wide variation, both in form and results obtained. Some of the factors are, strength of metal, shape of the teeth, point at which load is to be applied, and the velocity.

We will use the formula, H P = C V P F, in which H P equals the horse-power transmitted, C a constant or coefficient, P the pitch in inches, F the face of the tooth in inches and V the velocity in feet per minute.

Example. What horse-power can be transmitted by a gear whose pitch is 2 inches, face 5 inches, with a velocity of 700 feet per minute, assuming .003 as the constant.

Ans. H P = C V P F = .003 × 700 × 2 × 5 = 21 H. P.

For bevel gears, multiply the above result by $\frac{d}{D}$ or the ratio of the diameter of the pitch circles at the small end and at the large end, then,

$$H\ P = C\ V\ P\ F\ \frac{d}{D}$$

EXAMPLES FOR PRACTICE.

1. Neglecting clearance if the pitch is 3 inches, what is the addendum? the dedendum? Ans. { 1.05 inches, 1.20 inches.

2. Two wheels gear with each other. The driver has 42 teeth and makes 60 revolutions per minute. How many teeth has the follower, if it makes 36 revolutions per minute?

Ans. 70 teeth.

3. A follower has 13 teeth and makes 125 revolutions; find the number of teeth on the driver if it makes 25 revolutions.

Ans. 65 teeth.

4. How many teeth are there in a gear wheel whose pitch circle has a diameter of 13.92 inches and the pitch is $1\frac{1}{4}$ inches?

Ans. 35 teeth.

5. What is the diameter of a gear having 48 teeth and pitch of 1.75 inches? Ans. 26.73 inches.

6. There is a train of 4 gear wheels as in Figure 44. B and D are drivers and have 11 and 12 teeth respectively. If C has 48 teeth and A has 55 teeth, how many revolutions per minute will A make when D makes 700? Ans. 35 revolutions.

BELT GEARING.

A flexible connection used for the transmission of power is called belt, rope or chain gearing. In belt gearing the pulley is made either in one casting, or in several parts, bolted together. A pulley cast in two parts is shown in Fig. 45. The latter style is readily put on and removed from the shaft. On account of shrinkage strain in large castings, pulleys over six feet in diameter are usually cast in sections. In

order to prevent the belt from running off the pulley, the diameter at the centre of the face is generally made a little greater than that at the edges. This curved shape is called the crowning and the amount varies with the width of the face of the pulley. Single belts are usually made of one thickness of material. Double belting is made of two thicknesses cemented, riveted or laced together. Waterproof woven belting is extensively used, especially for great widths. Belts may be open or crossed as shown in Fig. 46. If the belts are open, the shafts revolve in the same direction, and in opposite direction, if crossed.

In practice, the usual way of determining the length of a belt, if the pulleys are in pos tion on the shafts, is to pass a tape around them. But for open running belts, not in position, the following formula, which gives approximate results, is used.

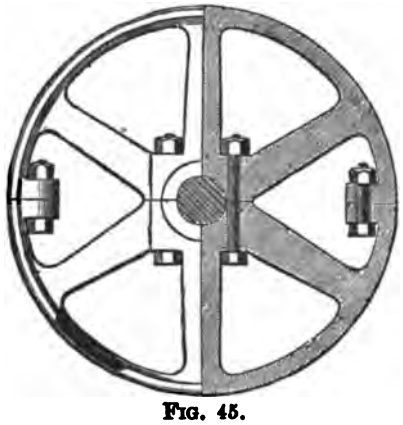


FIG. 45.

$$L = 3\tfrac{1}{4}\,\frac{(D + d)}{2} + 2\,l$$

When, L equals length of the belt in inches.

D equals diameter large pulley in inches.

d equals diameter small pulley in inches.

l equals distance between center of shafts in inches.

For example, the distance between the center of two pulleys is 10 feet; the diameters of the pulleys are 36 and 12 inches respectively. What is the length of the belt?

$$L = 3\tfrac{1}{4}\,\frac{(D + d)}{2} + 2\,l.$$

$$= 3\tfrac{1}{4}\,\frac{(36 + 12)}{2} + 2 \times 120.$$

$$= 318 \text{ inches or } 26\tfrac{1}{2} \text{ feet.}$$

If hemp rope is used, the rims are made with V shaped grooves, each rope having its own groove. The rope rests on the sides of the groove and is wedged between them so that the resistance to slipping is great. If wire rope is used, it rests on the

bottom of the groove, which is U shaped and is larger than the grooves for hemp rope.

Pulleys that rotate with great speed are balanced to prevent shaking and vibrating of the shaft under the influence of centrifugal force. If there were no slipping of the belt, the velocity of belt and the velocities of the surfaces of the pulleys would all be equal. In the consideration of pulleys, usually no allowance for slipping is made.

The half of the belt which is called the working side is stretched more tightly than the other, for if the tension on both sides were the same there would be no rotation.

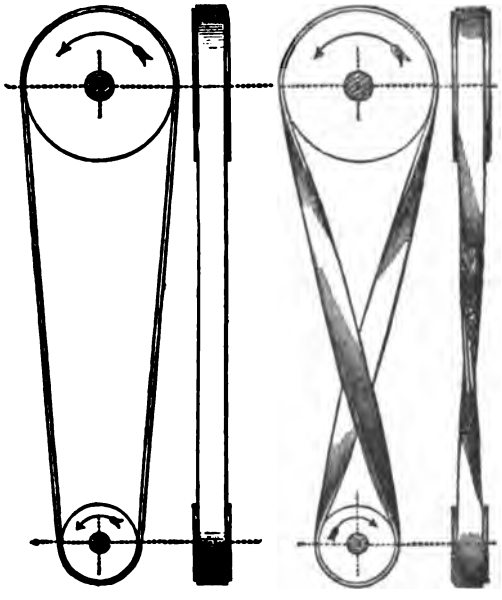


FIG. 46.

The calculation of speed and diameters of pulleys is similar to those of gears, except that the diameter, instead of the number of teeth, is used.

If a belt connects two pulleys A and B and the diameter of A is 18 inches and that of B is nine inches, B will make two revolutions while A makes one, or the revolutions of any two pulleys over which a belt runs, vary inversely as the diameters.

Let D equal the diameter of the driver.
Let *d* equal the diameter of the follower.
Let N equal the number of revolutions of the driver.
Let *n* equal the number of revolutions of the follower.
In all cases the product of the diameter and number of revo-

lutions of one equals the product of the diameter and number of revolutions of the other, or $D \times N = d \times n$.

Suppose a pulley 28 inches in diameter is driven by one 21 inches in diameter which makes 104 revolutions per minute, how many revolutions does the first make?

$$DN = dn. \quad n = \frac{DN}{d} = \frac{21 \times 104}{28} = 78 \text{ revolutions. Ans.}$$

In this manner, if any three terms are given, the other may be found from the formula

$$DN = dn, \quad D = \frac{d\ n}{N}, \quad N = \frac{d\ n}{D}, \quad d = \frac{DN}{n}, \quad n = \frac{DN}{d}.$$

Good leather belting should have considerable strength, but the working stress should not exceed 42 pounds per inch of width for single belts and 80 pounds for double belts.

The horse-power that a single belt will transmit is found by multiplying the width of the belt in inches by the velocity in feet per minute and dividing this product by 800, or, —

Let V equal velocity in feet per minute.

Let W equal width in inches.

$$H\ P = \frac{V\ W}{800}$$

For double belts this formula is

$$H\ P = \frac{VW}{540}$$

What horse-power can be transmitted by a single belt 8 inches wide, with 1500 feet per minute velocity?

$$H\ P = \frac{V\ W}{800} = \frac{1500 \times 8}{800} = 15\ H\ P. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

1. A pulley 6 inches in diameter is driven by another 21 inches in diameter, which makes 200 revolutions per minute. How many revolutions does the follower make? Ans. 700.

2. A pulley that makes 160 revolutions per minute is 18 inches in diameter; what must be the diameter of a pulley that is driven by the above, if it makes 60 revolutions per minute?

Ans. 48 inches.

3. A pulley of a 50 H. P. engine which makes 150 revolutions per minute is 4 feet in diameter; how wide must the single belt be? Ans. $21\frac{1}{4}$ inches.

4. If a 12 inch belt transmits 30 H. P., at what speed does it run? Ans. 2000 feet per minute.

5. What horse-power can be transmitted by a 24 inch single belt running over a pulley 4 feet in diameter that is making 200 revolutions per minute? If the belt is double?

75.398 + and 111.70 + H. P. Ans.

SECTION 5.

MECHANICS.

PART II.

COHESION AND ADHESION.

The force that holds together like molecules is called cohesion, while the force that holds together unlike molecules is called adhesion.

If a stick is whittled, a pencil sharpened, a board planed, or wood sawed, we find it necessary to use instruments, such as a knife, plane or saw. This is because the particles of wood are held together by a force called *cohesion*. It is very difficult to break a piece of iron because of the strong cohesive force of its particles.

The strength of tools and building material depends upon cohesive force. The supporting capacity of wire ropes, iron pillars, or steel rods is very great. Wood can be broken more readily than iron, because its particles have less cohesive force and are thus more easily separated. Liquids are far more easily separated. The hand can be thrust into water but not into wood, because the particles of a solid body are not so easily separated as those of a liquid.

The cohesion of liquids is exhibited in the formation of water globules when water is poured from a bottle drop by drop. In order to break a solid, the force of cohesion must be overcome by displacing its parts. The body is not really penetrated. When driving a nail into a board, the nail crowds the particles of the board aside and makes its way between them, for two bodies cannot occupy the same space at the same time.

If a solid is broken the cohesion between its particles at the break is overcome, and we cannot return this cohesive force. It is different with a liquid. Two parts can be readily placed together again. If a piece of iron is broken, it is impossible, by simply placing it together, to cause it to resume its former strength, but if it is heated sufficiently it approaches the liquid stage and the separated particles again cohere. This is shown in the process of welding.

It is evident, therefore, that solids have more cohesion than liquids, while gases have none. If a solid is heated, it is liquified (unless chemical separation takes place) and if the liquid thus formed is sufficiently heated, it is converted into a gas and its cohesion entirely destroyed.

Adhesion is the force that causes the crayon to leave traces upon the blackboard. This force acts between the molecules of chalk and the molecules of the blackboard, the result being that the particles of chalk remain upon the board.

If the hand be dipped in water it is wet when withdrawn. This is because the water adheres to the hand. If the hand be dipped in mercury, however, it is dry when withdrawn because the cohesive force of the mercury is greater than the adhesive force between it and the hand. When the hand is placed in water the particles of water next to it cling both to the hand and to the particles which are farther away, but they cling to the hand with greater force than they do to the other particles, hence the hand is wet.

In many cases both cohesion and adhesion act at the same time, producing important effects. For instance, when two pieces of wood are glued together the cohesion acts between the particles of the glue while adhesion acts between the particles of the wood and those of the glue, the effect being that the pieces of wood are strongly fastened together. Mortar, which is used in the construction of a brick wall, after hardening, causes the bricks to be held firmly in place. Cohesion acts between the particles of the mortar and adhesion acts between the particles of the mortar and those of the bricks.

CAPILLARITY.

If a glass plate be thrust into water, the water clings to the surface of the glass. This is due to the adhesion between the glass and the water. If this were not so the water would not cling to its surface.

If a glass rod be thrust into water, the water rises upon it, but if the glass rod be previously smeared with grease the surface of the water will be depressed. If a glass rod is placed in mercury, instead of rising the surface of the mercury will be depressed.

If, now, we take a glass tube and place it in water, the water will rise in it, as shown in Fig. 3, the surface of the water in the tube taking the form shown in Fig. 1. If, however, a tube be thrust into mercury, the mercury will be depressed, as shown in

FIG. 1.

FIG 2.

FIG. 3.

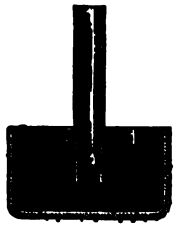


FIG. 4.

Fig. 4, the surface of the liquid in the tube taking the form shown in Fig. 2.

We know that water will wet the tube and that mercury will not, and we may now formulate the law, that if a liquid wets a surface it will rise upon it, but if it does not wet it, the liquid will be repelled.

In the case of thrusting a greased glass rod into water, the water was repelled because the water could not then wet the surface of the glass. The mercury was repelled for the same reason. The finer the bore of the tube, the higher the liquid, if it wets the tube, will ascend. In a tube $\frac{1}{100}$ of an inch in diameter, water will rise over 5 inches.

This capillary attraction, as it is called, is the result of adhesion operating between solid and liquid bodies. If a lump of sugar be touched to water, the sugar will become wet throughout. The same thing occurs in the case of a piece of blotting paper or a sponge. Lamp wicks draw up the oil to the flame, and grease spots may often be removed by placing them in contact with earth. Clothes become wet from the rain. In short, almost everything about us is made up of capillary tubes.

STRENGTH OF MATERIALS.

If a force is applied to any body, it tends to change the shape or volume of that body; for example, if a pressure is applied to a

gas, the volume is decreased, that is, its molecules are forced more closely together. If the pressure is reduced the volume is increased, *i. e.*, the molecules move further apart than before. With gases, a slight pressure causes considerable change of volume, while in liquids and solids, the pressure must be very great in order to produce a perceptible change. If a bar of iron is fixed firmly at its upper end and a heavy weight is hung from the lower, the bar lengthens. If a weight is placed on the top of a block of wood or metal the block is shortened. In the above examples, the weight or force tends to change the shape or volume or both, and the forces within the body that tend to resist the change are called *stresses*. The word "*stress*" is also used as a name for the force. Engineers say that a body is strained or stressed when a load is placed upon it.

The weights or forces which act externally are also called stresses. Stresses, like forces, are measured in pounds, tons, grams, kilograms, etc. For example, if a weight of fifteen hundred pounds is suspended by a rope, the stress in the rope is fifteen hundred pounds. If a block of iron has a weight or force of forty tons on its top, the stress is forty tons. If the force tends to pull the particles of the body apart, it is called *tensile stress*, if it tends to crush the body, it is called *compressive stress*. The stress that resists two forces which tend to cut a body between them is called a *shearing stress*. There are other stresses; transverse, seen in the bending of beams, and torsional, a force that resists twisting, the propeller shaft of a marine engine, for instance. Of these stresses, Tensile, Compressive, and Shearing are called the simple stresses and are the only ones that will be considered here.

A Unit Stress, or unit strain, is the amount of stress on a unit area of section and is expressed as pounds per square inch, tons per square inch or per square foot, kilograms per square centimeter, etc. For example, if a block of cast iron has an area of three square inches and a stress of 15,000 pounds is applied to the upper surface, the stress per square inch will be $\frac{15,000}{3} =$ 5,000 pounds per square inch. Let A equal the area in square inches, P the total stress in pounds, then the unit stress will be $\frac{P}{A}$. If S equals unit stress, we have the formulas,

$$\frac{P}{A} = S, \quad P = AS, \quad A = \frac{P}{S}.$$

Suppose a brick in a testing machine to crush at a stress of ten tons, the section of the brick being 2×4 inches. The unit compressive strain is

$$\frac{P}{A} = \frac{20{,}000}{8} = 2{,}500 \text{ pounds.}$$

If a rope supports a weight of 314.16 pounds, the unit tensile strain being 400 pounds per square inch, what is the diameter of the rope?

$$A = \frac{P}{S} = \frac{314.16}{400} = .7854 \text{ square inches.}$$

Since the area is .7854 square inches, the diameter is one inch.

When a force or weight is applied to a bar or block, there is a change of form or volume. If the stress is tensile, the body lengthens, if it is compressive, the body shortens. This change of shape is called deformation and is measured in inches, centimeters, etc. This change of shape is also called *strain*. Strain is, however, an ambiguous term, being used by some as stress and by others as deformation. If we use the term strain, explanation will be given as to the meaning intended, by use of the terms *stress* or *deformation*. *Unit deformation* is the amount of lengthening or shortening per unit of length. Let L equal length in feet, b the elongation or shortening per unit of length, and B the total deformation, then evidently $L \times b = B$, or $B = bL$. If a bar of iron ten feet long lengthens .1 of an inch under a load of 20,000 pounds, the elongation per foot will be

$$b = \frac{B}{L} = \frac{.1}{10} = .01 \text{ inches.}$$

If a small load is placed on a body, a slight change of shape will take place; if the load is removed the body will regain its original shape. This quality or property of returning to original shape on release of load is called *elasticity*.

Experiment shows that deformation is approximately proportional to the load, that is, within the *elastic limit*. If the load is increased, the deformation is also increased, that is, if a bar lengthens a given amount under a stress of 2,000 pounds, it will lengthen twice as much under a stress of 4,000 pounds and three

times as much under 6,000 pounds. Deformation is also approximately proportional to length. If a block 6 inches long shortens .02 of an inch under 50,000 pounds, a similar block 12 inches long will shorten .04 of an inch under the same load.

If the stress is great and on removal, the body does not return to its original size and shape, the body has been deformed beyond its elastic limit and has been given a permanent set, or in other words the cohesive force of the molecules has been partially overcome. When the load is increased until the piece breaks, crushes, or shears, the breaking stress is reached and the force per unit of area applied at the moment of rupture is called the *ultimate breaking strength.*

Many experiments and tests have been made to determine the action and strength of different materials under various tensile and compressive stresses. The usual method is to place them in a testing machine and while the load is being gradually applied to carefully measure the deformation. By noting the load at the moment of rupture, the breaking strength is determined. By means of these experiments, together with experience, the following general laws applying to simple compression and tension have been established and may be taken as fundamental principles.

1. When a small stress is applied to a body, a small deformation is produced and on removal of the stress, the body returns to its original form. Materials may be regarded as perfectly elastic for small stresses.

2. Under small stresses, the deformations are nearly proportional to those stresses, and also approximately proportional to the length of the body.

3. When the stress is sufficiently great, a deformation is produced which is partly permanent, that is, the body does not spring back to its original shape when the stress or strain is removed. This permanent change of shape is called *set* or *permanent set.* In such cases, deformation is not proportional to stress. When the elastic limit is exceeded, deformation is not proportional to stress.

4. If the stress is still further increased, deformation increases until the piece breaks.

5. A sudden stress or shock is more injurious than a steady stress gradually applied.

The words small and great in the above laws have different values for different materials, that is, a large stress for wood would not be a large stress for steel.

The following table gives approximate constants for those materials in most common use:

	Coefficient of Elasticity.	Elastic Limit.	Ultimate Tensile Strength.	Ultimate Compressive Strength.	Ultimate Shearing Strength.
Steel	30,000,000	50,000	100,000	150,000	70,000
Wro't Iron	25,000,000	25,000	55,000	55,000	50,000
Cast Iron	15,000,000	6,000	20,000	90,000	20,000
Stone	6,000,000			6,000	
Brick				2,500	
Timber	1,500,000	3,000	10,000	8,000	

ELASTICITY.

According to the first law given above, materials are perfectly elastic for small stresses. When the load is large and the piece does not return to its original shape, it is said to have a permanent set. The unit stress at which permanent set is first visible is called the elastic limit.

The body being perfectly elastic within the elastic limit, laws describing its action can be formulated, but beyond the elastic limit, there being a permanent alteration of shape, these laws cannot be applied.

In testing a specimen of any material, it is easy to reduce its stress to unit stress by the formula $S = \frac{P}{A}$ and its deformation to unit deformation by the formula $b = \frac{B}{L}$. Suppose the unit stress and the unit deformation are known, the ratio of these is the *Coefficient of Elasticity*, that is, the ratio of the unit stress to the unit strain or deformation is the Coefficient of Elasticity or, expressed algebraically

$$\frac{S}{b} = E = \textit{Coefficient of Elasticity.}$$

For example, suppose the unit stress is 45,000 pounds and the unit deformation is .0015 inch. What is E?

$$E = \frac{S}{b} = \frac{45{,}000}{.0015} = 30{,}000{,}000.$$

A flat cast iron foundation ring, four inches high, whose area is 4 square feet, has a weight of 144 tons placed on the top. If the weight causes a shortening of .00016 of an inch, what is the Coefficient of Elasticity?

$$E = \frac{S}{b}, \quad S = \frac{P}{A}, \quad b = \frac{B}{L}.$$

Then substitute,

$$E = \frac{PL}{AB} = \frac{144 \times 2000 \times 4}{4 \times 144 \times .00016} = 12{,}500{,}000.$$

The Coefficient of Elasticity is also expressed, in the case of tension, as the unit stress which would elongate a bar to double its original length, provided this could be done without exceeding the elastic limit.

EXAMPLES FOR PRACTICE.

1. If a bar of steel is 8 inches long and 1 inch in diameter, and breaks under a load of 70,000 pounds, what is the unit breaking stress? Ans. 89,000 pounds (approx.)

2. If stone weighs 160 pounds per cubic foot and its ultimate compressive strength is 6000 pounds, how high must a tower be to crush under its own weight? Ans. 5409 feet.

3. Find the diameter of a round wrought iron bar, to stand a tensile stress of fifty tons. Breaking stress 50,000 pounds per square inch. Ans. 1.6— inches.

4. A tensile stress of 4500 pounds is applied to a wrought iron bar $\frac{3}{4}$ inch in diameter. Find the unit stress.

Ans. 10,000 pounds (approx.)

5. Find the probable elongation of a wooden bar nine inches square and 12 feet long, under a tensile stress of 9000 pounds if the unit elongation is .008 inch. Ans. .096 inch.

6. A brick 2 × 4 × 8 inches is under a compressive stress of 24,000 pounds. If this stress is distributed uniformly over the surface, which is 2 × 4 inches, what is the unit stress?

Ans. 3000 pounds.

7. A bar 3 inches in diameter and 9 feet long, elongates .12 inch under a load of 30,000 pounds. How much will a bar 3 inches in diameter and 18 feet long elongate under a stress of 90,000 pounds? Ans. .72 inches.

8. Find the diameter of a short steel piston rod, when the piston is 16 inches in diameter, the steam pressure is 75 pounds per square inch and the safe working pressure is 8000 pounds per square inch. Ans. $1\frac{9}{16}$ inches (nearly.)

TENSION.

Tensile stress in a body is the resistance offered by its molecules to being pulled apart. If we apply a gradually increasing load to a bar, it shows the following changes. While the unit stress is less than the elastic limit, the unit elongation or unit strain is small and proportional to the unit stress. The elongation is also proportional to the length. Within the elastic limit, the ratio of unit stress to unit strain or deformation is the coefficient of elasticity.

After the elastic limit is passed, the bar elongates rapidly at its weakest point, and the area of the cross section is considerably reduced. When the stress is increased sufficiently, the bar tears apart or breaks. In most cases, the breaking stress is the maximum unit stress, but sometimes the unit stress is maximum before rupture, that is, some materials after being strained nearly to the breaking stress, will break at much less than the maximum stress. The ductility of a material is judged by the elongation and the reduction of area of the cross section. Those materials which have slight elongation and but little reduction in area are not very ductile. Some varieties of steel and most grades of cast iron, have very slight reduction of area of cross section. The usual length of a test piece is eight times its diameter.

COMPRESSION.

The compressive or crushing strength of any material is the resistance offered by its molecules to being pushed nearer together. It is the opposite of tension. Within the elastic limit, the piece shortens in proportion to the stress. After passing the elastic

limit, the amount of shortening increases rapidly until the piece crushes or splinters. An enlargement of cross section accompanies shortening. The length of test pieces should never exceed five times their diameter or they will fail by a combination of stresses, which is made up of compressive and bending stresses. The coefficients of elasticity, and elastic limits for steel, wrought iron, and wood are usually given the same as those calculated by tensile tests, but many believe the average values of coefficients of elasticity and elastic limit are greater for compression than for tension.

SHEAR.

The shearing strength of a material is the resistance offered by its fibres to being cut or, if not fibrous, the resistance offered by its molecules to being slipped by each other. It is somewhat different from tension and compression. Shear occurs when two forces tend to cut a body between them. The punching of a plate is an example of this stress. When a bolt is in tension, the stress tends to shear off the head and threads. When two parallel plates transmit tension, the plates tend to shear the rivet which connects them.

Like tension and compression, the unit shearing stress is found from the formula $S = \frac{P}{A}$, in which A is the sheared surface. The ultimate shearing strength is the stress P, which causes rupture, divided by the area. The elastic limit and the amount of deformation or detrusion are difficult to determine accurately but approximate values are found by the testing machine.

TORSION.

A torsional stress occurs when forces tend to twist a body about its axis. This stress will be discussed under "Transmission of Power by Shafting."

FACTOR OF SAFETY.

In order that no part of a machine or building may change its shape and so destroy the machine or structure, it must not be strained beyond the elastic limit of its materials nor should it be

strained so as to have permanent set. Parts of machines should be designed so that they will not only stand all stresses applied to them without breaking, but they should never be stressed beyond the elastic limit. In order to comply with the above, it is usual to design the piece large enough so that the greatest stress will be some fraction of the breaking stress. This *working stress*, as it is called, varies with the material and the use to be made of the same.

If the piston rod of an engine has a maximum unit stress of 7800 pounds and we know that the metal will stand 85,800 pounds per square inch, then the ratio of the ultimate stress to the working stress is $\frac{85800}{7800} = 11$. This result is called the factor of safety, and is the number of times that the ultimate strength is greater than the working stress, or it is the ratio of the ultimate strength to the working stress. Let S_t be the ultimate strength, S_w the working stress, and f the factor of safety, then

$$f = \frac{S_t}{S_w} \text{ or } S_w = \frac{S_t}{f}$$

The factors of safety used in American practice are given in the following table:

Material	For Steady Stresses.	For Varying Stresses.	For Shocks (Machines).
Steel	5	7	15
Wrought Iron	4	6	10
Cast Iron	6	15	20
Brick and Stone	15	25	30
Timber	8	10	15

EXAMPLES FOR PRACTICE.

1. The ultimate strength of a piece of mild steel is 64,000 pounds per square inch. What load will be supported by a bar 2 inches in diameter, using a factor of safety of 6?

Ans. 33,500 pounds (approximately).

2. Find the factor of safety when 90,000 pounds are placed on a block of cast iron whose area is $2 \times 1\frac{1}{2}$ inches, ultimate strength being 90,000 pounds per square inch. Ans. 3.

3. A stay rod of a boiler has a stress of 7000 pounds. If it is $1\frac{1}{8}$ inches in diameter and a factor of safety of 10 is used, what is the breaking strength? Ans. 70.400 pounds (approximately).

STRENGTH OF PIPES AND CYLINDERS.

Water and steam exert equal pressures in all directions and tend to rupture a pipe longitudinally or in the direction of its axis. The material resists this rupturing stress.

Let p = the pressure per square inch.
D = diameter of the pipe.
L = length of the pipe.
t = thickness of the pipe.
S = tensile stress.

Then, the force which tends to cause rupture is equal to the pressure on a plane perpendicular to the direction of pressure. The plane of projection is equal to the projection of the semi-cylindrical surface, or it is equal to the diameter multiplied by the length. The force then, is equal to the diameter multiplied by the length, multiplied by the pressure per square inch, or, Force = p LD.

The resistance of each side is equal to the area of metal, multiplied by the stress per square inch, or tLS, and for two sides, $2t$LS. Then, as the resistance must be at least equal to the force tending to rupture

$$p\,\mathrm{LD} = 2\,t\,\mathrm{LS}$$

$$\text{or,}\quad p\,\mathrm{D} = 2\,t\,\mathrm{S}$$

Dividing by 2, the formula becomes $\frac{p\mathrm{D}}{2} = t\mathrm{S}$ or, since $\mathrm{D} = 2r$; $pr = t\mathrm{S}$ and the equation may be written as,—

$$\frac{t}{r} = \frac{p}{\mathrm{S}};\ \text{or}\ \frac{\text{thickness}}{\text{radius}} = \frac{\text{pressure per square inch}}{\text{working stress}}.$$

In the solution of problems, the above ratio will be found to be a convenient one.

For example: What should be the thickness of a cast iron pipe ten inches in diameter, with an internal pressure of 150 pounds per square inch, and 1,250 pounds per square inch as the safe working stress? Solution —

$$\frac{t}{r} = \frac{p}{\mathrm{S}}\quad \text{or,}\quad t = \frac{r\,p}{\mathrm{S}} = \frac{5 \times 150}{1250} = .6$$

The pipe should therefore be about $\frac{5}{8}$ of an inch thick.

If head in feet be given the pressure per square inch in the

pipe can be found, by multiplying the head by .434, the formula being: $P = .434\ h$ as is explained under Hydrostatics.

What should be the thickness of a cast iron water pipe with an ultimate strength of 20,000 pounds per square inch, if it is sixteen inches in diameter, when the head is 300 feet, and a factor of safety of 15 is used?

$$t = \frac{r\,p}{S}. \qquad S = \frac{20000}{15} = 1333 \text{ lbs.}$$

$$p = .434 \times 300 = 130.2, \quad r = 8. \quad t = \frac{8 \times 130.2}{1333} = .78 \text{ in.}$$

A thin cylinder will tend to fail longitudinally as will a pipe, and the force tending to rupture it transversely is found by multiplying the area of the head of the cylinder by the pressure per square inch, or $\frac{1}{4}\,\pi\,D^2 \times p$.

The resistance of the metal is equal to the area of cross-section of the cylinder multiplied by its unit stress or, $\pi\,Dt \times S$.

Equating these two formulas,

$$\frac{1}{4}\,\pi\,D^2 p = \pi\,Dt\,S.$$

$$\frac{Dp}{4} = t\,S$$

$$Dp = 4t\,S,$$

or the resistance to transverse rupture is twice the resistance to longitudinal rupture.

A thin sphere subjected to internal pressure tends to rupture in the line of a great circle, the formula being the same as for transverse rupture of the cylinder, $Dp = 4\,t\,S$.

Cylinders under external pressure are not treated as those under internal pressure. The condition would be the same if the cylinder remained a true circle in cross section, but the external pressure tends to increase any slight variation from the true circle and make it elliptical. Cylinders or tubes fail by collapsing rather than from crushing.

We can calculate the strength of thin cylinders under external pressure only by empirical formulas deduced from results of experiments. Fairbain gives the formula,

$$P = \frac{806000\,t^{2.19}}{l\,d}$$

in which t is the thickness in inches, l is the length in feet, d the diameter in inches, and P the collapsing pressure in pounds per square inch. A similar formula,

$$P = \frac{9600000\ t^{2.18}}{l\ d}$$

is given by Wood, the letters standing for the same quantities, except that l is inches instead of feet. In these two formulas t^2 in place of $t^{2.18}$ or $t^{2.19}$ is accurate enough for most calculations.

What is the proper thickness of a wrought iron tube or cylinder, diameter four inches, length 72 inches or six feet, pressure 120 pounds per square inch, using a factor of safety of ten?

$$P = \frac{9600000t^2}{ld},\quad t^2 = \frac{P \times l \times d}{9600000},\quad t^2 = \frac{120 \times 72 \times 4 \times 10}{9600000}$$

$= .036$ inches and $t = \sqrt{.036} = .19$ inches.

EXAMPLES FOR PRACTICE.

1. What should be the thickness of a wrought iron pipe four inches in diameter, with an internal pressure of 140 pounds per square inch, if 4000 pounds per square inch is the allowable working pressure? Ans. .07 inch.

2. Find the pressure head in a cast iron pipe 20 inches in diameter, when the thickness is $\frac{3}{4}$ inches and the working stress is 1666 pounds per square inch. Ans. 287. + feet.

3. Find the factor of safety used in the case of a cast iron water pipe 12 inches in diameter and $\frac{5}{8}$ inch thick which is under a head of 300 feet, the ultimate strength being 20,000 pounds per square inch. Ans. Nearly 16.

4. What should be the thickness of a cast iron sphere three feet in diameter, if the internal pressure is 180 pounds per square inch, with fifteen as a factor of safety? Ultimate strength 20,000 pounds per square inch. Ans. 1.21 + inches.

5. If the above sphere were $\frac{1}{2}$ inch thick, at what internal pressure would it burst, assuming the cast iron to rupture at 18,000 pounds per square inch. Ans. 1000 pounds per square inch.

6. Find t from the formula,

$$t^2 = \frac{P \times l \times d}{9600000}$$

when the cylinder is 60 inches long, 6 inches in diameter, and the

external pressure is 130 pounds per square inch, using 10 as the factor of safety. Ans. .22 inches.

BEAMS.

A simple beam is a bar resting at its ends on supports. Simple beams are of three classes: (*1*) Those with both ends fixed; (*2*) With one end fixed and one end simply supported; (*3*) With both ends simply supported. Beams may be fixed by building their ends into walls or masonry.

A beam fixed at one end and having the other end unsupported is called a cantilever beam.

The loads on beams are either uniform, or concentrated at some point or points. Any load evenly spread on a beam is called uniform. The weight of the beam itself is a uniform load.

Concentrated loads are those that are taken as if they acted at a single given point.

Uniform loads are expressed in pounds per linear foot. Then if a beam four feet long weighs twelve pounds per foot and has a uniform load of ten pounds per foot, the total load will be $(4 \times 12) + (10 \times 4) = 88$ pounds. If this beam were supported at the two ends, it is evident that each support would react upward an amount equal to $\frac{1}{2}$ the sum of the weight and load, or 44 pounds each. These upward forces are called the reactions of the supports.

Since the beam is in equilibrium, the upward reactions must equal the loads acting downward. If the sum of all the vertical forces equals zero, and the sum of the moments of all forces equals zero, then the beam will be in equilibrium. From the first condition, the sum of all the loads on the beam must equal the sum of the reactions. If there is but one support, as in the case of the cantilever beam, there is but one reaction and it is equal to the sum of the loads. If there are two supports, the second condition must be used with the first. To find the moments, the centre of moments may be taken anywhere in the plane of the beam, but it is convenient to take it at one of the supports.

Suppose we have a beam, as shown in Fig. 5. The beam is 12 feet between supports and we will consider it as without weight. A load of ten pounds acts downward three feet from the

right hand reaction, R_2. Find the reactions. Let us take moments about the left hand support R_1, then $12\ R_2 - 9 \times 10 = 0$ or $R_2 =$ $7\frac{1}{2}$ lbs. and if moments are taken about R_2, $12\ R_1 - 3 \times 10 = 0$ or $R_1 = 2\frac{1}{2}$ lbs. and $R_2 = 7\frac{1}{2}$ lbs. A check on the result is $R_1 + R_2 =$ loads or $2\frac{1}{2}$ lbs. $+ 7\frac{1}{2}$ lbs. $= 10$ lbs., which shows the above to be correct.

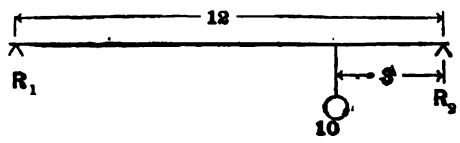


FIG. 5.

Let us take another example. A simple beam, Fig. 6, 18 feet long, weighs 30 pounds per linear foot. A weight of 700 pounds is placed 5 feet from the left end and one of 500 pounds 8 feet from the right end. Find the reactions.

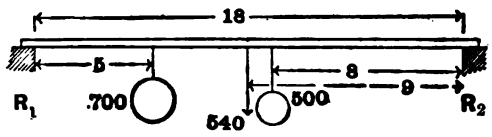


FIG. 6.

If the beam weighs 30 pounds per foot, the total weight of the beam will be $30 \times 18 = 540$ pounds, and since it is uniformly distributed, it may be considered as acting at the centre, or 9 feet from either end. Taking the centre of moments at R_1, we have the moments of forces $R_2 \times 18 = (540 \times 9) + (5 \times 700) + (10 \times 500)$ from which $R_2 = 742\frac{2}{9}$ pounds and taking moments about R_2, $R_1 \times 18 = (540 \times 9) + (500 \times 8) + (700 \times 13)$. $R_1 = 997\frac{7}{9}$ pounds. Adding R_1 and R_2 we find $742\frac{2}{9} + 997\frac{7}{9} = 1740$ pounds, which is equal to the sum of the loads.

In case there are more than two supports, the beam is continuous and the reactions cannot be found without knowing the elastic properties of the material.

Beams usually fail by breaking across *i. e.* by transverse rupture. From a consideration of forces and moments together with the use of higher mathematics, the following simple formulas have been deduced for the designing of beams.

1. For a cantilever loaded at the end with a load W,

$$W = \frac{S}{l} \times \frac{I}{c}$$

2. For a cantilever uniformly loaded with a load W,

$$W = 2\frac{S}{l} \times \frac{I}{c}$$

3. For a simple beam loaded at the middle with a load W,

$$W = 4\frac{S}{l} \times \frac{I}{c}$$

4. For a simple beam uniformly loaded with a load W,

$$W = 8\frac{S}{l} \times \frac{I}{c}$$

In the above formulas, W equals the load in pounds. S equals safe working stress of the material in pounds per square inch, l equals length in inches, I equals moment of inertia of the section, c equals the distance of the fibre most remote from the neutral axis. The neutral axis being a line which does not change in length when the beam is bent.

The following is a table of the values of I and c for most common forms.

For a rectangle with base b and depth d,

$$I = \frac{b\,d^3}{12}, \quad c = \frac{d}{2}$$

For a circle with diameter d,

$$I = \frac{\pi d^4}{64}, \quad c = \frac{d}{2}$$

For a triangle of base b and depth d,

$$I = \frac{bd^3}{36}, \quad c = \frac{2d}{3}$$

For a square with side d, and one diagonal vertical,

$$I = \frac{d^4}{12}, \quad c = .707d$$

For I beams with depth d, I, * $c = \frac{1}{2}d$.

For example, a rectangular wooden beam of 14 feet span carries a load of 1000 pounds at the middle. Find the depth when its width is 4 inches and S = 800 pounds.

$$W = 4\frac{S}{l} \times \frac{I}{c}$$

$$1000 = \frac{4 \times 800}{14 \times 12} \times \frac{I}{c} \text{ therefore } \frac{I}{c} = 52.5$$

from the table,

$$I = \frac{bd^3}{12} = \frac{4d^3}{12} = \frac{d^3}{3} \text{ and } c = \frac{d}{2}$$

* See table of I beams, page 20.

hence,

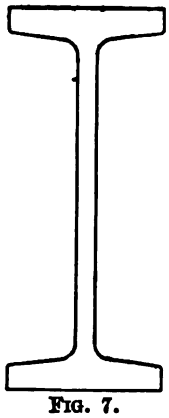


FIG. 7.

$$\frac{I}{c} = \frac{d^3}{3} \times \frac{2}{d} = \frac{2d^2}{3} = 52.5$$

from which $d^2 = 78.75$ and $d = 8.87$ inches.

Wrought iron I beams, Fig. 7, are rolled in different sizes a few of which are given in the following table. They are used in engineering and architectural work.

The area of the cross section of any beam in the table may be approximately obtained by multiplying its weight per foot by 3 and dividing by 10, since a wrought iron bar 3 feet long, and one square inch in section weighs 10 pounds.

TABLE OF WROUGHT IRON I BEAMS.

Depth in Inches.	Width of Flange. Inches.	Weight per foot Pounds.	Value of $\frac{I}{c}$	Value of I	Least value of I
Heavy 15	5.81	80	100	750	29.9
Light 15	5.55	67	90.3	677	25.4
Heavy 12	5.09	60	56.7	340	15.5
Light 12	4.64	42	45.9	275	11.0
Heavy 10	4.77	45	37.5	187	11.3
Light 10	4.32	30	30.0	150	7.94
Heavy 9	4.33	33	26.0	117	7.14
Light 9	4.01	23.5	21.7	97.5	5.48
Heavy 8	4.29	35	22.6	90.4	6.96
Light 8	3.81	22	17.5	69.6	4.57
Heavy 7	3.91	25	15.5	54.3	4.87
Light 7	3.61	18	13.1	45.8	3.72
Heavy 6	3.46	18	9.48	28.4	2.51
Light 6	3.24	13.5	8.16	24.5	2.00
Heavy 5	2.91	13	5.69	14.2	1.34
Light 5	2.73	10	4.94	12.3	1.08
Heavy 4	2.63	10	3.50	6.99	.87
Light 4	2.48	8	3.10	6.19	.71
Heavy 3	2.52	9	2.36	3.54	.84
Light 3	2.32	7	2.06	3.09	.55

A wrought iron I beam of 12 feet span supports a uniform load of 55,000 pounds. If a factor of safety of 5 be used, what is the size of the I beam?

It is evident that

$$S = \frac{55{,}000}{5}$$

$$\text{and, } W = 8\,\frac{S}{l} \times \frac{I}{c}$$

$$55{,}000 = 8\,\frac{\frac{55{,}000}{5}}{12 \times 12} \times \frac{I}{c}$$

$$\text{Hence, } \frac{I}{c} = 90$$

From the table $\frac{I}{c}$ for a light 15 inch beam is 90.3, and as it fulfils the conditions it should be used.

Deck beams, Fig. 8, are also extensively used. The head is formed of arcs of circles, the flange being similar to that of an I beam. Fig. 9 shows sections of a few other forms in which wrought iron beams are rolled. The section represented by Z is called a Z bar, by T called a Tee section bar, by S a bulb angle bar, and by U a channel bar, and by N an angle bar. The following table

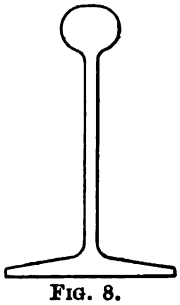


FIG. 8.

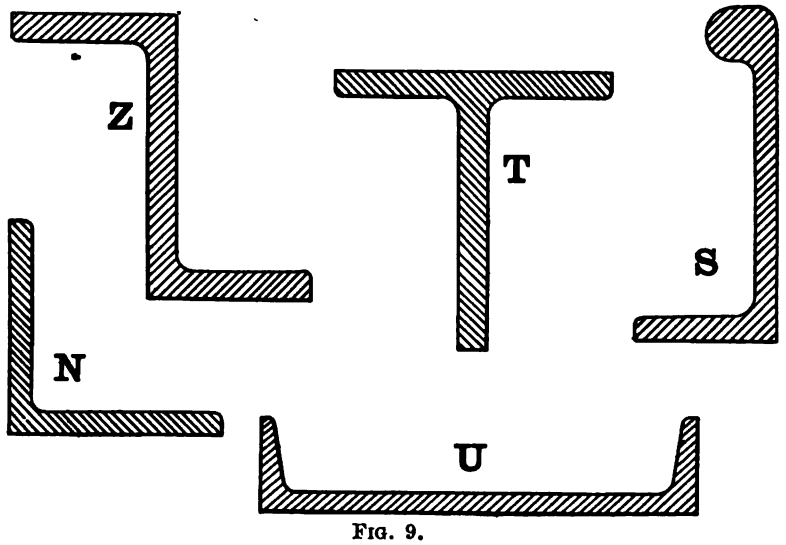


FIG. 9.

gives some of the more important dimensions and values of $\frac{I}{c}$.

TABLE OF DECK BEAMS.

	Depth. Inches.	Width of Flange. Inches.	Weight per foot. Pounds.	Value of $\frac{I}{c}$
Heavy	9	3.97	30	20.0
Light	9	3.75	23.5	17.1
Heavy	8	4.00	28	14.1
Light	8	3.75	21.5	11.6
Heavy	7	3.75	23	10.8
Light	7	3.50	17	8.6

A deck beam of 10 feet span is loaded uniformly with 15,000 pounds. If 12,000 pounds is the allowable stress, what size beam will be selected?

$$W = 8\frac{S}{l} \times \frac{I}{c}; \quad 15{,}000 = 8 \times \frac{12{,}000}{10 \times 12} \times \frac{I}{c}; \quad \frac{I}{c} = 18.75$$

From the table a heavy 9 inch deck beam would be selected.

If a light 8 inch deck beam was loaded uniformly with 50,000 pounds and is of 12 foot span, would it safely carry the load?

$$W = 8 \times \frac{S}{l} \times \frac{I}{c}; \quad 50{,}000 = 8 \times \frac{S}{12 \times 12} \times 11.6$$

from which S equals nearly 77,600 pounds and as an average breaking strength for wrought iron is 55,000 pounds, it would not carry the load.

A heavy 9 inch I beam used as a cantilever, is 10 feet long with a load at its end of 3000 pounds. What is the factor of safety, if the weight of the beam is neglected?

For a cantilever loaded at the end, with a load W

$$W = \frac{SI}{lc} \quad \text{or,} \quad 3000 = \frac{S}{10 \times 12} \times 26.$$

$$S = \frac{3000 \times 12 \times 10}{26} = 13{,}846 \text{ pounds.}$$

If 13,846 pounds is the safe working stress, $\frac{55{,}000}{13{,}846}$ is the factor of safety = 3.97 or about 4.

EXAMPLES FOR PRACTICE.

1. A beam 9 feet long, weighing 20 pounds per foot, has a load of 200 pounds in the middle. What is the reaction of the support at each end? Ans. 190 pounds.

2. If the above beam has, in addition, a load of 150 pounds, 3 feet from the left end, what are the reactions?

Ans. 240 pounds at one and 290 pounds at the other support.

3. A square wooden beam 10 feet long has a uniform load of 500 pounds per linear foot. If the working stress is 500 pounds, what is a side of its cross-section, neglecting the weight of the beam? Ans. 9.65 inches.

4. A cast iron beam 1 inch in diameter has a load of 25 pounds at the middle. If the stress is 2000 pounds per square inch, what is the length? Ans. 2.6 + feet.

5. A wrought iron beam, 2 inches square and 8 feet long, has a total uniform load of 1200 pounds, when used as a cantilever. What is the working stress?

Ans. 43,200 pounds per sq. in.

6. A heavy 6 inch wrought iron I beam has a uniform load of 500 pounds per foot. What is the factor of safety, when the span is 12 feet and the beam weighs 18 pounds per foot? Take 55,000 pounds as the ultimate strength. Ans. 4.66.

7. What I beam will be selected when the span is $8\frac{1}{2}$ feet, the load 10,000 pounds at the middle, the working stress 9000 pounds, neglecting the weight of the beam?

Ans. Light 10 inch beam.

8. What is the uniform load on a heavy 8 inch deck beam, 11 feet long, with a factor of safety of 5? Take 55,000 pounds as the ultimate strength. Ans. $854\frac{6}{11}$ pounds per foot.

Exterior forces, or loads, cause beams to bend. The amount of bending depends on the load, the material and the dimensions of the beam. In common structures a very slight deflection is not serious but most parts, especially of machinery, must be stiff enough to prevent perceptible bending. The stiffness of a beam is indicated by the load it can carry with a given deflection.

Mathematical equations and experiments show that deflections vary directly as the load W, directly as the cube of the length, inversely as the coefficient of elasticity, inversely as the moment of inertia and inversely as the cube of the depth.

Stiffness varies directly as the coefficient of elasticity, directly as the moment of inertia and inversely as the cube of the length.

The following table shows the values of the maximum deflections of a few simple cases, with the relative deflection and stiffness.

	Max. deflection.	Relative deflection.	Relative strength.	Relative stiffness.
Cantilever loaded at end,	$\frac{1}{3}\frac{Wl^3}{EI}$	1	1	1
Cantilever loaded uniformly,	$\frac{1}{8}\frac{Wl^3}{EI}$	$\frac{3}{8}$	2	$2\frac{2}{3}$
Simple beam loaded at middle,	$\frac{1}{48}\frac{Wl^3}{EI}$	$\frac{1}{16}$	4	16
Simple beam uniformly loaded,	$\frac{5}{384}\frac{Wl^3}{EI}$	$\frac{5}{128}$	8	$25\frac{3}{5}$

For example: Two beams have the same width, length and load; the first has a depth of 2 inches and the second a depth of 4 inches. Compare their relative deflections.

Since the deflection of rectangular beams varies inversely as the cube of the depth, the deflections will vary as 4^3 to 2^3 or 64 to 8, or 8 to 1. Therefore the first beam will deflect 8 times as much as the second.

A wooden beam 4 inches square, 10 feet long, is supported at both ends and is loaded with 800 pounds at the middle. What is the maximum deflection? Note — (Reduce length to inches).

The maximum deflection equals

$$\frac{1}{48}\frac{Wl^3}{EI} \text{ or } \frac{1 \times 800 \times (12 \times 10)^3}{48 \times 1{,}500{,}000 \times I}.$$

$$I = \frac{d^4}{12} = \frac{4^4}{12} = \frac{256}{12} = \frac{64}{3}$$

then inserting the value for I the quantity becomes

$$\frac{1 \times 800 \times (12 \times 10)^3 \times 3}{48 \times 1{,}500{,}000 \times 64} = .9 \text{ inch}$$

Therefore the beam deflects .9 inch.

If the beam had been uniformly loaded with 80 pounds per foot, how would its relative strength, deflection and stiffness compare with the above.

The relative strength would be as 8 to 4, therefore the uniformly loaded beam would be stressed only one-half as much.

From the table we find that the relative deflection of a simple

beam loaded at the middle is $\frac{1}{16}$, and of a simple beam uniformly loaded is $\frac{5}{128}$. Hence the beam with a uniform load will be deflected in the ratio of $\frac{5}{128}$ to $\frac{1}{16}$ which may be expressed as $\frac{5}{8}$. The maximum deflection of a simple beam loaded at the middle has already been determined; it is .9 inch. Thus the deflection of the uniformly loaded beam must be $\frac{5}{8}$ of $\frac{9}{10}$ or $\frac{9}{16}$ inch.

The relative stiffness would be as $25\frac{3}{5}$ to 16 or 8 to 5.

Find the deflection of a heavy 7 inch I beam, as a cantilever 6 feet long loaded at the end with 1 ton.

The formula from the table is $\frac{1}{3}\frac{Wl^3}{EI}$.

I from the table is 54.3

$$\frac{1}{3}\frac{Wl^3}{EI} = \frac{1}{3}\frac{2000\ (12 \times 6)^3}{25{,}000{,}000 \times 54.3} = .183 \text{ inch.}$$

EXAMPLES FOR PRACTICE.

1. Compare the deflection of a simple rectangular beam 2 inches wide and 4 inches deep with that of a simple beam having the same load 2 inches square and $\frac{1}{2}$ the length of the first, neglecting the weight of the beam. Ans. Same deflection.

2. Compare the stiffness of a beam 3 × 8 inches when laid with the 3 inch side vertical, and when laid with the 8 inch side vertical. Ans. 9 : 64.

3. Find the deflection of a simple rectangular wrought iron beam 9 feet long 2 × 3 inches in section with the 3 inch side vertical, when there is a uniform load of 100 pounds per foot.

Ans. .131 + inch.

4. How much would the beam have been deflected if the entire load of 1080 pounds had been hung in the center?

Ans. .25 + inch.

5. What is the deflection of a cast iron cantilever beam 2 × 2 inches and 6 feet long with a load of 100 pounds at the end?

.17 + inch due to its own weight
.62 + inch due to load at end
Total .79 + inch.

COLUMNS.

Columns are prisms, with a length of more than ten times the least diameter. If the prism is very short it fails by simple compression as we have already seen; if it is long it fails by bending

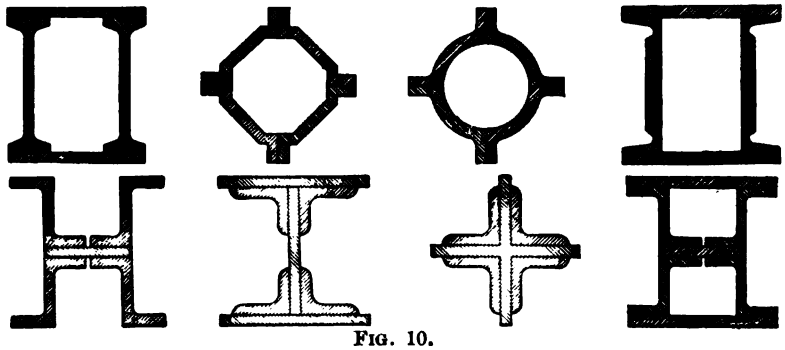


FIG. 10.

sideways. There are many kinds and shapes of columns. They are built round, square, hexagonal, etc., and are both solid and hollow. If of wood they are usually round or square. Cast iron columns are generally round. The wrought iron column is made in a great variety of forms. They are made of I beams, or of different shaped irons riveted together as shown in Fig. 10. Columns are made symmetrical so that the liability to bending will be the same for all directions. The shape of the ends makes considerable difference as to the strength. The ends are either round or fixed. In Fig. 11, R has rounded ends, H one end rounded and the other fixed, and F has both ends fixed. Columns with both ends fixed are the most used, but in bridge construction and machines columns with rounded ends are frequently used.

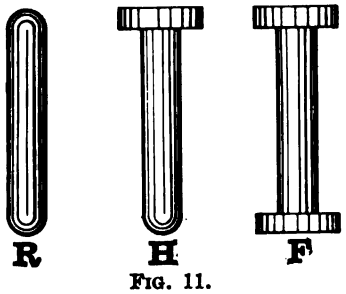


FIG. 11.

Experiments show that columns with fixed ends are stronger than with rounded ends, so the order of strength of the above columns is F, H, R. And with the same lengths, sizes and material, the relative strengths are R, 1; H, 2 $\frac{1}{4}$ and F, 4. Other things being equal a column with both ends fixed is 4 times as strong as one with both ends rounded.

The mathematical formulas for columns are empirical owing to the imperfect tests made upon them.

A formula deduced by Euler, applicable only to columns very long in proportion to their diameter, is:

$$P = C \frac{\pi^2 E I}{l^2}$$

in which C = 1, $\frac{2}{4}$, or 4 for the columns R, H and F respectively, E = coefficient of elasticity, I = moment of Inertia, P = weight, l = length in inches.

Find the weight on a rectangular wooden column 4 × 6 inches and 24 feet long, ends fixed.

$$P = \frac{4 \times 9.8696 \times 1{,}500{,}000 \times I}{288 \times 288}$$

$$I = \frac{bd^3}{12} = \frac{4 \times 6^3}{12} = 72, \text{ I also } = \frac{bd^3}{12} = \frac{6 \times 4^3}{12} = 32$$

We use the least moment of inertia since the column will bend in the direction of the least resistance; then substituting we have,

$$P = \frac{4 \times 9.8696 \times 1{,}500{,}000 \times 32}{288 \times 288} = 22{,}843 \text{ pounds}$$

or 951 pounds per square inch.

The formula

$$P = C \frac{\pi^2 E I}{l^2}$$

may be written

$$P = \frac{C \pi^2 E b d^3}{12 l^2}$$

for a rectangular column, d, being the least side or,

$$P = \frac{C \pi^3 E d^4}{64 l^2}$$

for a cylindrical column of diameter d. If r be the least radius of gyration of the cross section, the formula may be written,

$$\frac{P}{A} = \frac{C \pi^2 E r^2}{l^2}$$

where C = 1, $2\frac{1}{4}$ or 4 as before.

The value P above is not the breaking strength but the load which causes bending. This load is the one we want, for the

bending practically marks the beginning of failure. The above formulas show that the strength of a column varies directly as its cross section and directly as the square of the least diameter or least side.

The following table gives values of r^2 for the most used forms.

For a rectangle, least side d	$r^2 = \frac{d^2}{12}$
For a circle of diameter d	$r^2 = \frac{d^2}{16}$
For a triangle whose least altitude is d	$r^2 = \frac{d^2}{18}$
For a hollow square section	$r^2 = \frac{D^2 + d^2}{12}$
For a hollow circular section	$r^2 = \frac{D^2 + d^2}{16}$

For I beams, deck beams and other shapes, r^2 is the least moment of inertia divided by the area of cross section. Both these quantities may be found from tables or calculated.

A formula for ultimate strength of wrought iron columns such as are ordinarily used in practice is:

$$S = \frac{40{,}000}{1 + \frac{(12\,l)^2}{n\,r^2}}$$

in which S = the ultimate strength, l = the length in feet, r^2 = the value of r^2 from the table, n = 36,000 for columns with fixed ends, 24,000 for columns with one end fixed and the other round, 18,000 for both ends round. Factors of safety of 4 or 5 should be used with the above formula.

EXAMPLES FOR PRACTICE.

1. What is the stress on a wrought iron column 12 feet high, when $r^2 = .50$. Use the formula

$$S = \frac{40{,}000}{1 + \frac{(12\,l)^2}{n\,r^2}};$$

when n = 36,000 and a factor of safety of 4 is used.

Ans. 4,647 pounds (nearly).

2. The length of a wrought iron column fixed at both ends is 12 feet. Find the diameter when a factor of safety of 4 is used and the stress is 8700 pounds.

Ans. 8 inches (approx.)

TRANSMISSION OF POWER BY SHAFTING.

Shafting is often the means by which power is transmitted from place to place. Shafts are usually made of wrought iron or steel in the form of a cylindrical bar. Cold rolled shafting is made by a special rolling process. If turned in a lathe it is called bright shafting. Black shafts are of bar iron rolled by the usual process and machined wherever bearings are placed.

It is evident that the most important resistance which a shaft must overcome in transmitting power acts to prevent the shaft from turning. Hence the strength of material with regard to the amount of torsion it can withstand must be known before we choose the material for a shaft. As has been stated, torsion occurs when applied forces tend to twist a body.

The cause of torsion in shafts is due to the stress produced by pulleys, belts, ropes, etc. The twisting of a main shaft is due to the power of the engine applied at the crank. The torsion set up in the large shafts of our ocean steamships is enormous. Hence an accurate knowledge of the value of torsion under different conditions is important. Shafts are subjected to a bending action due to their own weight and to the weight of the pulleys, belts, etc., which they support. As this bending action is difficult to determine without the aid of higher mathematics, we will consider torsion only.

Suppose one end of a horizontal shaft to be rigidly fixed, and let the free end have a lever p attached at right angles to its axis, as shown in Fig. 12. If a weight A be hung at the end of this lever, the shaft will be twisted so that a fibre such as is represented by the dotted line ae will

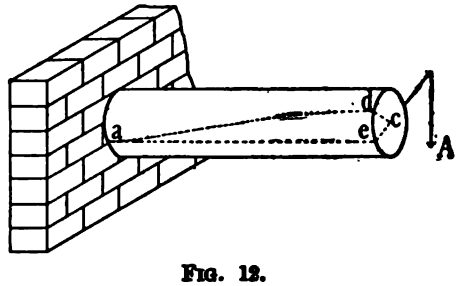


FIG. 12.

assume some such spiral form as *ad*. The end of the shaft will also be found to have rotated through a certain angle *d c e*.

It can be shown that if A is not so large as to strain the material beyond its elastic limit, the angles *dce* and *ead* are proportional to the weight applied, and that when this weight is removed the lines *dc* and *da* return to their former positions *ec* and *ea*. If the elastic limit is exceeded and the twisting is great enough the shaft will be ruptured.

Let us represent the perpendicular distance from the axis of the shaft to the line of action of A by p, then the moment of A, which tends to twist the shaft with regard to the axis must be Ap. This product is called the twisting moment, and whatever the number of forces acting at the end of the shaft their resultant moment may always be represented by Ap. We may now consider a simple illustrative example. A force of 80 pounds at 18 inches from the axis twists a shaft 60°. What force will produce the same result when acting 4 feet from the axis? We have seen that Ap is a constant quantity. Hence we may state the equation:

$$Ap = xp'$$

$$\text{or} \quad x = \frac{Ap}{p'},$$

substituting the proper values this becomes

$$x = \frac{80 \times 18}{4 \times 12}, \text{ and } x = 30 \text{ pounds.}$$

EXAMPLE FOR PRACTICE.

A force of 200 pounds acting at a distance of 6 inches from the axis, twists a shaft 2 feet long through an angle of 7 degrees. Through what angle will a similar shaft 4 feet long be twisted by a force of 500 pounds acting at a distance of 18 inches from the axis? Ans. 105 degrees.

When a shaft is designed to transmit power, the ratio of the greatest internal shearing unit stress S_s to the power H, must be known. Otherwise the shaft might be ruptured by too great a load. Let us consider a shaft making n revolutions per minute, which transmits H horse-power. The work may be applied by a belt from the motor to a pulley on the shaft. By virtue of the

resistance of the material and its elasticity, the shaft carries the work to other pulleys and belts. These in turn transmit to the working machines.

The torsion S_s to which the shaft is subjected, evidently increases with the power transmitted, H. Let P be the resistance acting at the circumference of the pulley, the radius of which is r. For each revolution the force P must act through the circumference $2\pi r$. Hence the work performed in one revolution must be $2\pi r$ P, and in n revolutions it performs the work $2\pi r$ P n.

If P is expressed in pounds and r in inches, the transmitted horse-power is:

$$H = \frac{2\pi r P n}{33000 \times 12} \qquad (1.)$$

The twisting moment P r in this expression may be expressed by the resisting moment, $\frac{S_s J}{c}$, in which J is the polar moment of inertia and c is the distance from the axis about which twisting occurs, to the remotest part of the cross section. The axis in the above instance is usually at the center of figure of cross section. Substituting this value of P r in the above equation it becomes:

$$H = \frac{\pi n S_s J}{198000 c} \qquad (2.)$$

This formula is the more convenient form since its factors are readily derived. J and c must be taken in inches, S_s in pounds per square inch, and n is the number of revolutions per minute. The following is a table of values of J and c for some of the most common cases.

For a circle with a diameter d,

$$J = \frac{\pi d^4}{32}; \quad c = \frac{d}{2}$$

For a square whose side is d,

$$J = \frac{d^4}{6}; \quad c = d\sqrt{\frac{1}{2}}$$

For a rectangle with sides b and d,

$$J = \frac{b d^3}{12} + \frac{b^3 d}{12}; \quad c = \frac{1}{2}\sqrt{b^2 + d^2}$$

The following values are Constants of Torsion, and have been computed from experiments on the rupture of shafts.

For timber $S_s = 2{,}000$ pounds per square inch.
For cast iron $S_s = 25{,}000$ pounds per square inch.
For wrought iron $S_s = 50{,}000$ pounds per square inch.
For steel $S_s = 75{,}000$ pounds per square inch.

A wooden shaft 6 inches square breaks when making 40 revolutions per minute. Find the horse-power then probably transmitted. Let us use the equation

$$H = \frac{\pi n S_s J}{198000 c}$$

If we substitute the proper values this becomes:

$$H = \frac{3.1416 \times 40 \times 2000 \times 6^4}{198000 \times 6 \sqrt{\frac{1}{2}} \times 6}, \text{ and } H = 64 \text{ (approx.)}$$

A steel shaft 3 inches square makes 10 revolutions per minute. Find the H. P. under which it will probably break.

Ans. 76 H. P. (approx.)

The diameter of a round shaft may readily be found from equation (2). If we insert for J and c their proper values and solve for d we obtain;

$$d = 68.5 \sqrt[3]{\frac{H}{n S_s}} \qquad (3.)$$

The degree of security of the shaft can be found by comparing computed values of S_s with those already given. The computed values of S_s should be obtained from the formula,

$$S_s = 321000 \frac{H}{n d^3}$$

Find the diameter of a wrought iron shaft which will transmit 90 H. P. when the number of revolutions is 100 per minute, using a factor of safety of 8. Substitute the proper values in equation (3) then:

$$d = 68.5 \sqrt[3]{\frac{90}{100 \frac{50000}{8}}}$$

or, $d = 3.58$ inches (approx.)

EXAMPLES FOR PRACTICE.

1. Find the factor of safety for a wrought iron shaft $2\frac{1}{2}$ inches in diameter when transmitting 25 H. P. and while making 100 revolutions per minute. Ans. 9.7 (approx.)

2. What should be the diameter of a wrought iron shaft to transmit 90 horse-power with a factor of safety of 8 when making 250 revolutions per minute? Ans. 2.65 inches. (approx.)

METRIC SYSTEM.

The French system of measurements is based upon a unit of length called a *meter*. This was intended to be one forty millionth of a great circle passing around the globe. Multiples of this number are obtained by the Greek prefixes, deka-10, hekto-100, kilo-1,000, and myria-10,000. Thus a dekameter means 10 meters, a hektometer 100 meters, a kilometer 1,000 meters, and a myriameter 10,000 meters.

The meter is divided into smaller divisions which are designated by means of the Latin prefixes, deci-$\frac{1}{10}$, centi-$\frac{1}{100}$, milli-$\frac{1}{1000}$. Thus a decimeter is $\frac{1}{10}$ of a meter, a centimeter $\frac{1}{100}$ of a meter and a millimeter $\frac{1}{1000}$ of a meter.

LINEAR MEASURE.

10 millimeters (*mm.*)	= 1 centimeter (*cm.*)
10 centimeters	= 1 decimeter (*dm.*)
10 decimeters	= 1 meter (*m.*)
10 meters	= 1 dekameter (*Dm.*)
10 dekameters	= 1 hektometer (*Hm.*)
10 hektometers	= 1 kilometer (*Km.*)
10 kilometers	= 1 myriameter (*Mm.*)

SQUARE MEASURE.

100 square millimeters (*sq. mm.*)	= 1 square centimeter (*sq. cm.*)
100 square centimeters	= 1 square decimeter (*sq. dm.*)
100 square decimeters	= 1 square meter (*sq. m*) = 1 centar (*ca.*)
100 square meters	= 1 square dekameter = 1 ar (*a.*)
100 square dekameters	= 1 square hektometer = 1 hektar (*Ha.*)
100 square hektometers	= 1 square kilometer (*sq. Km.*)

CUBIC MEASURE.

1000 cubic millimeters (*cu. mm.*)	= 1 cubic centimeter (*cu. cm.*)
1000 cubic centimeters	= 1 cubic decimeter (*cu. dm.*) = 1 liter.
1000 cubic decimeters	= 1 cubic meter (*cu. m.*)

WEIGHTS.

The unit of weight in the metric system is the *gram*. A gram is the weight of one cubic centimeter of pure water at the temperature of 4 degrees centigrade (39.2 degrees Fahrenheit) at which temperature water has its greatest density. Larger and smaller weights are derived from the gram.

WEIGHTS.

10 milligrams (*mg.*)	= 1 centigram (*cg.*).	
10 centigrams	= 1 decigram (*dg.*).	
10 decigrams	= 1 gram (*g.*)	= wt. of 1 cu. cm. of water.
10 grams	= 1 dekagram (*Dg.*).	
10 dekagrams	= 1 hektogram (*Hg.*).	
10 hektograms	= 1 kilogram (*K.*)	= wt. of 1 cu. dm. of water.
10 kilograms	= 1 myriagram (*Mg.*).	
10 myriagrams	= 1 quintal (*Q.*).	
10 quintals	= 1 metric ton (*T.*)	= wt. of 1 cu. m. of water.

TABLE OF EQUIVALENTS.

1 inch = 2.540 centimeters.	1 centimeter = 0.3937 inch.
1 foot = 3.048 decimeters.	1 decimeter = 0.328 foot.
1 yard = 0.9144 meter.	1 meter = 1.0936 yds. = 39.37 in.
1 sq. inch = 6.452 sq. centimeters.	1 sq. centimeter = 0.1550 sq. inch.
1 sq. foot = 9.2903 sq. decimeters.	1 sq. decimeter = 0.1076 sq. foot.
1 sq. yard = 0.8361 sq. meter.	1 sq. meter = 1.196 sq. yards.
1 cu. inch = 16.387 cu. centimeters.	1 cu. centimeter = 0.0610 cu. inch.
1 cu. foot = 28.317 cu. decimeters.	1 cu. decimeter = 0.0353 cu. foot.
1 cu. yard = 0.7645 cu. meter.	1 cu. meter = 1.308 cu. yards.
1 ounce av. = 28.35 grams.	1 gram = 0.03527 ounce av.
1 pound av. = 0.4536 kilogram.	1 kilogram = 2.2046 pounds av.
1 ton (2,000 lbs.) = 0.9072 met. ton.	1 metric ton = 1.1023 tons.

When changing units of one denomination to an equivalent number of units of another denomination it is only necessary to move the decimal point as follows:

(1) In linear measure, when changing units to the next higher or lower denomination, move the decimal point *one* place to the left or right respectively.

(2) In square measure, when changing units to the next higher or lower denomination, move the decimal point *two* places to the left or right respectively.

(3) In cubic measure, when changing units to the next higher or lower denomination, move the decimal point *three* places to the left or right respectively.

By paragraph (1) it will be seen that in linear measure if the unit is to be changed to the second higher or lower denomination, the decimal point must be moved *two* places, or if to the third higher or lower denomination, *three* places, etc.

By paragraph (2) it is apparent that in square measure if the unit is to be changed to the second higher or lower denomination, the decimal point must be moved *four* places, or if to the third higher or lower denomination, *six* places, etc.

By paragraph (3) may be seen that in cubic measure if the unit is to be changed to the second higher or lower denomination, the decimal point must be moved *six* places, or if to the third higher or lower denomination, *nine* places, etc.

EXAMPLES FOR PRACTICE.

1. Change 1 meter to decimeters. Ans. 10 dm.
2. Change 1145 céntimeters to dekameters. Ans. 1.145 Dm.
3. Change 98 square dekameters to square meters. Ans. 9800 sq. m.
4. Change 11,125 square dekameters to square kilometers. Ans. 1.1125 sq. Km.
5. Change 1 cubic meter to liters. Ans. 1000 liters.
6. Change 11,254,734,368 cubic millimeters to cubic meters. Ans. 11.254734368 cu. m.
7. How many meters are there in a bar which is 9.8425 yards long? Ans. 9 meters.
8. The distance from A to B is 262.4667 feet. How many decimeters in the same distance? Ans. 800 decimeters.
9. The bed of an engine has a surface area of 538.19 square feet. How many square meters does the same bed contain? Ans. 50 square meters.
10. How many cubic decimeters are there in a pipe containing 176.572 cubic feet? Ans. 5000 cubic decimeters.
11. A cylinder contains 95.101 quarts when filled. What is its capacity in liters? Ans. 90 liters.
12. The capacity of a box is 237.753 gallons. Find its capacity in liters. Ans. 900 liters.
13. A safety valve weighs 17.63697 pounds. Find its weight in kilograms. Ans. 8 kilograms.

HYDROSTATICS.

If several vessels containing water are connected by tubes, the water stands at the same level in all of them, unless prevented by some other force, and if prevented from doing this the water exerts an upward pressure. In fact, in liquids the same pressure is exerted in every direction at any given point. The greater the depth of the liquid the greater is the pressure.

Take, for instance, the pressure on the bases of the three vessels shown in Fig 13. If the base of every one of these vessels has the same area, the pressure upon it is the same as upon the others, *i. e.*, the pressure is proportional to the depth. In the first case this pressure is the weight of the entire column of water resting upon VQ. In the second case, it is the weight of the column VKIQ resting upon VQ, while the small side columns are supported by the shelves OH and LN. In the third case, however, the pressure on VQ is the same because the water exerts an upward pressure against the shelves OH and LN, which is equivalent to columns of water included by the dotted lines and the sides of the vessel, and this upward pressure causes a reaction against the base VQ of the vessel, so that we see pressure is proportional to depth and not to the amount of liquid. The pressure upon VQ in the third case is greater than the entire weight of water contained in the vessel.

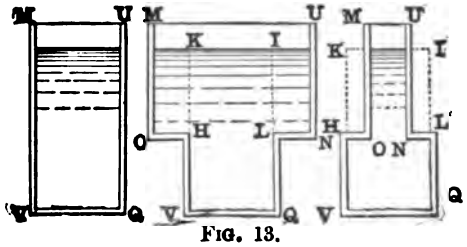


FIG. 13.

If we have a pail containing water and a hole be made in one side of it, the water flows out because the reaction against the pressure of the water is removed. We will investigate quantitatively these laws of pressure a little later.

DENSITY.

Some bodies are heavier than others, *i. e.*, of two bodies of the same size but different in substance, one weighs more than the other. Suppose a cubic foot of a substance to weigh 115 pounds and a cubic foot of another to weigh 80 pounds. These bodies are

of the same size but the former weighs more. This is due to their difference of density. The density of the former is 115 pounds per cubic foot, and of the latter 80 pounds per cubic foot.

Thus we express density by the number of units of weight in some unit of volume. Water has a density of about $62\frac{1}{2}$ pounds per cubic foot. If we use the French system we may say that a body weighs a certain number of grams per cubic centimeter. A convenient way of expressing density is by comparing the density of a substance with that of water, as is discussed under the next heading.

SPECIFIC GRAVITY.

The specific gravity of a body is the ratio between its weight and the weight of a like volume of some other substance taken as a standard. The standard which is adopted for solids and liquids is distilled water at a temperature of 4 degrees C or 39.2 degrees F.

For aeriform bodies the ordinary standard is hydrogen. In order to obtain the specific gravity of a substance we must divide the weight of a given volume of it by the weight of the same volume of water. We find the weight of an equal bulk of water, which is the divisor, in various ways, but the weight of the substance is always used as the dividend. We will now proceed to discuss the various methods of determining this ratio.

First, let us consider the case of determining the specific gravity of a fluid such as oil by means of the specific gravity bottle. Suppose that the bottle is weighed. Suppose, also, that the bottle is weighed when full of water and when full of oil. If we subtract the weight of the bottle from the weights obtained in each of these cases we have the weights of equal bulks or volumes of the oil and of the water, and if we divide the former by the latter we obtain the specific gravity of the former, *i. e.*, the density of the oil compared with that of water taken as a standard.

Let us now take a case of a solid which will sink in water. It is heavier than water, and therefore its specific gravity must be greater than unity. If we have a vessel containing water filled to the point of overflowing and then place the solid in that vessel, a certain amount of water equal in volume to the solid, overflows. If we weigh this amount of water which is run over, we have the

weight of an amount of water equal in bulk to the substance which we placed in the water, and if we divide the weight of the substance by the weight of this amount of water we obtain as a result the specific gravity of the substance. There is an easier way of finding the specific gravity of a substance which sinks in water. Suppose we weigh the substance in the air. Now if we weigh it in water we find that it loses in weight and that *this loss of weight is equal to the weight of the water which would overflow on placing the solid in a vessel of water filled to overflowing.* Hence the loss of weight in water is equal to the weight of an equal bulk of water, and if we divide the weight of the substance by this loss of weight in water we obtain the specific gravity. If we represent the weight of the substance in air by W and the weight of the substance in water by W′ we obtain the following formula.

$$\text{Specific gravity} = \frac{W}{W - W'}$$

A body weighs 6 pounds in the air and 3.5 pounds in water; what is its specific gravity?

Solution,—

Substitute in the above formula the values given in the example, thus,—

$$\text{Specific gravity} = \frac{6}{6 - 3.5}$$

$$\text{or Sp. gr.} = 2.4. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE.

1. A body weighs 8 pounds in the air and loses $2\frac{1}{2}$ pounds when weighed in water, what is its specific gravity? Ans. 3.2.

2. Find the specific gravity of a body which weighs 120 pounds in air and 24 pounds in water. Ans. $1\frac{1}{4}$.

If we have a substance which is lighter than water and wish to determine its specific gravity, we cannot use the preceding method because the substance does not sink but floats upon the water, *i. e.*, its specific gravity is less than unity. If we place it in a vessel which is filled to the point of overflowing, a weight of water runs over precisely equivalent to the weight of the substance. As the substance floats part way out of water, we see that an equal volume of water is not displaced, but we may obtain the specific grav-

ity if we know how much of the substance is submerged in the water, *i. e.*, we may divide the volume of the water displaced by the volume of the substance because weight is inversely proportional to volume.

What is the specific gravity of a piece of wood which floats with two-ninths of its volume above the surface of water?

Solution,—

We consider the volume of wood as $\frac{9}{9}$ and of this $\frac{7}{9}$ is submerged. Therefore, to find the specific gravity we divide 7 which represents the volume submerged, *i. e.*, the volume of an equal weight of water by 9 which represents the total volume, and thus obtain for our result $\frac{7}{9} = .778 —$.

EXAMPLES FOR PRACTICE.

1. The volume of an air-tight cylinder is 946. When placed in water the volume of the part above the surface is 236.5. Find its Sp. Gr. Ans. .75.

2. What is the specific gravity of a body which floats in water with four-fifths of its volume submerged? Ans. .8.

It is often very inconvenient to use the above method, and when occasion demands, it is often convenient to use a form of the preceding method. This is as follows:

We ascertain the weight of a substance in air and we also weigh a sinker suspended in the water. We now attach the sinker to the substance and we find that together they weigh less than did the sinker alone, *i. e.*, the substance on being forced down into the water by the weight of the sinker, has exerted an upward force which makes the force exerted by the sinker on the spring balance (or whatever we use in weighing) less than before. Hence to find the specific gravity by this method we divide the the weight of the substance by its loss of weight in water, which is equivalent to its own weight plus the loss of weight of the sinker when attached to it. This is illustrated by the following example.

A body weighs 4 pounds in air. The sinker weighs 6 pounds in water and 3 pounds when attached to the body. Find the specific gravity of the body. Solution — Divide the weight of the body 4, by its loss of weight in water or $4 + (6 — 3) = 7$ and thus obtain $\frac{4}{7} = .571 +$ answer.

EXAMPLE FOR PRACTICE.

1. A body weighs 36 ounces in air. When attached to the body the sinker loses in weight 28 ounces. What is the specific gravity of the body? Answer .5625

Specific gravity may be also determined by means of instruments called hydrometers. These do not require the very careful handling that the balance does, though they are not as accurate. They are of two kinds, —

1. Hydrometers of constant volume.
2. Hydrometers of constant weight.

The loss of weight in water, which we have just discussed, is a very familiar fact. Under water a man can carry a large stone which would be difficult to lift in air. Timber when submerged has a negative weight, that is, it tends to rise to the surface. We have seen that this loss of weight of a submerged body is equal to the weight of an equal volume of water. When any body floats upon water the amount of water displaced is exactly equal to its own weight.

Let us consider a little further the construction and use of the hydrometer, taking for our first example the one known as the Nicholson. This instrument consists of a hollow cylinder with a weighted basket C, Fig. 14, at its lower end. The basket must be sufficiently heavy to keep the instrument vertical when floating in the water. A vertical rod with a pan A, for holding weights rises from the top of the cylinder. The whole apparatus is sufficiently light to float in the water and to sustain a certain weight W which causes it to sink to a certain point *o*, which is marked on the rod. The weight of the body whose specific gravity we wish to determine must be less than W otherwise it would sink the instrument. To ascertain the specific gravity we place the given body in the pan and add enough weights to submerge the instrument to the point *o*. Let *w* represent the weights required. The weight of the body is, obviously, W — *w*. Take

FIG. 14.

the body from the pan and place it in the basket. Then it will be found necessary in order to sink the instrument to the same point o, to add more weights to make up for the water displaced by the substance in the basket. Let x represent the weight thus required. The specific gravity must be the weight of the substance W — w, divided by the weight of the water displaced, which we have found to be x, or

$$\text{Specific gravity} = \frac{W - w}{x}$$

Another form of hydrometer which may be used to determine the specfic gravity of a liquid is the Fahrenheit. It is made of glass instead of metal and has a bulb loaded with shot or mercury in place of the basket. The weight W of the instrument is accurately determined. If the instrument is now placed in water a weight w must be added before the glass will sink to the zero point. Thus the weight of water displaced is W + w. If we now change the hydrometer to the liquid the specific gravity of which we wish to know, we must place a certain weight x in the pan in order to sink the instrument to the zero point. In this case the weight of the liquid displaced is W + x. Hence;

$$\text{Specific Gravity} = \frac{W + x}{W + w}$$

The results obtained by these hydrometers are not as accurate as those obtained by the methods previously given.

FIG. 15.

Let us now take a case of a constant weight hydrometer. This consists of a glass tube, Fig. 15 having, near one end, two bulbs. The smaller one which is nearer the end, is filled with some heavy substance. The other bulb and the tube contain air, and thus prevent the instrument from sinking. The loaded end compels an upright position. The point to which the hydrometer sinks in pure water is generally marked zero on the tube, which is then marked off or graduated, above and below. Two instru-

ments are sometimes used, one with the zero point near the top, for use in liquids heavier than water; and the other with the zero point near the bulb, for use in liquids lighter than water.

This instrument is not as frequently used to obtain exact specific gravities, as to determine the degree of concentration or dilution of certain liquids such as acids, alcohols, milk, sugar, solutions, etc. In all cases the principle utilized is that a floating body will displace a weight of the liquid in which it floats, precisely equal to its own weight. A greater volume of light than of heavy liquids, will of course, be displaced by the given weight.

PRESSURES.

Water is practically incompressible and has a remarkable capacity for transmitting a pressure unchanged in intensity in all directions. This is a consequence of the perfect fluidity of the water by which its particles move freely over each other and thus transmit the applied pressure. If we apply a pressure at some point of the surface of a closed vessel, that same pressure unchanged in intensity, is exerted over the entire inner surface of the vessel. If by means of a force pump we exerted a pressure of 10 pounds per square inch at the bung hole of a cask, that same pressure of 10 pounds per square inch would be exerted at every point of the inner surface of the cask. This principle is taken advantage of in the hydrostatic press.

Suppose, Fig. 16, that a pressure of 50 pounds is exerted on the small plunger S. Supposing that the area of the hole through which the plunger passes is 1 square inch, then that pressure is transmitted over to the other part of the press and upward against whatever may be in the press. If the area of the large plunger is 2 square feet, the total upward pressure would be the number of square inches, 288 multiplied by the pounds per square inch, 50, and the result would thus be 14,400 pounds. If the area of the large plunger were twice as great, *i. e.*, 4 square feet, the total upward pressure would be 28,800 pounds. What is here gained in pressure is lost in velocity, because if the area of the large plunger is 288 times that of the smaller the distance it moves for a given distance traversed by the small plunger is only $\frac{1}{288}$ as much. What is gained in force is lost in velocity. The pressure

exerted at any point within the volume of a body is, as we have seen, the same in every direction. If the water is deep the pressure increases.

If we have a barrel with a long tube running up from the

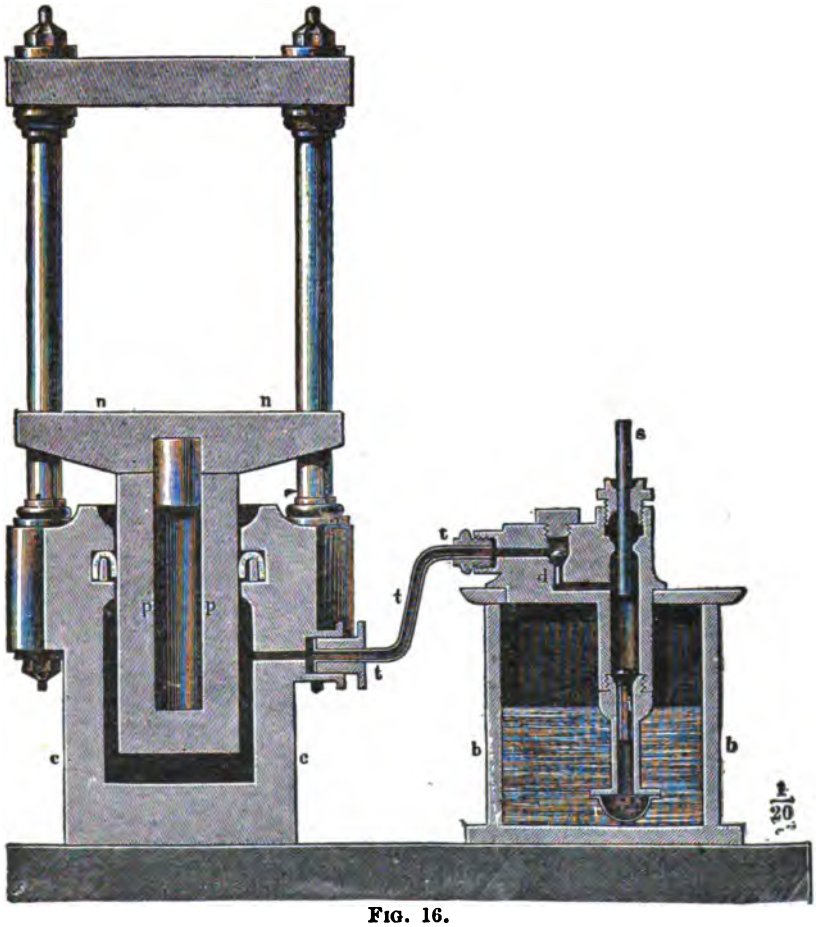


FIG. 16.

head of it and the whole filled with water, the depth of water may be sufficient to force the barrel apart. The free surface of water at rest is perpendicular to the direction of gravity and for bodies of water of small extent may be regarded as a plane. Any depth below this plane is called "a head." That is, the head upon any point is its vertical depth below the level surface.

Let h represent the head and w the weight of a cubic unit of water. Then at the depth h one horizontal square unit has a pressure equal to the weight of a column of water whose height is h and whose cross section is one square unit, but the pressure at this point is exerted in all directions with equal intensity. Therefore, the unit of pressure p at the depth h is wh. We may write this in a formula as follows;

$$p = wh.$$

Conversely the depth or head for the unit of pressure p may be written

$$h = \frac{p}{w}$$

If h is taken in feet and p in pounds for a square foot we obtain the following formulas;

$$p = 62.5\ h \text{ and}$$

$$h = \frac{p}{62.5} = .016\ p$$

because the weight of a cubic foot of water is 62.5 and we have taken the cubic foot of water as our unit. Thus we see that pressure and head are mutually convertible. They are often used as synonymous terms, although in reality each is proportional to the other.

Any pressure p may be regarded as produced by a head h which in turn is regarded as the pressure head. Numerically p is usually taken in pounds per square inch, while h is expressed in feet. Thus, since the pressure in pounds per square foot is 62.5 h, the pressure in pounds per square inch is $\frac{1}{144}$ of this, or

$$p = .434\ h \text{ and}$$
$$h = 2.304\ p.$$

To state these in words one foot head corresponds to. 434 pounds per square inch, and one pound per square inch corresponds to

2.304 feet head. But it must be remembered that these values depend upon the assumption that water weighs 62.5 pounds per cubic foot, and are therefore liable to variation in the third decimal place. If it is necessary to obtain the absolute pressure we must add atmospheric pressure which is usually about 14.7 pounds per square inch. Atmospheric pressure will be discussed later.

The relation between head and pressure will perhaps be seen more plainly by supposing the case of a cubic foot of water in a vessel, the area of the bottom of which is one square foot. Then we have the weight of a column one foot in height having a base area of one square foot. As the total weight is 62.5 pounds, the weight on each of the 144 square inches composing the square foot is equal to $\frac{62.5}{144} = .434$ pounds. Table I. gives the pressure in pounds per square inch for heads from 1 to 10 feet, and the head in feet corresponding to pressures of from 1 to 10 pounds per square inch, considering water to weigh 62.5 and 62.3 pounds per cubic foot.

TABLE I. HEAD AND PRESSURES.

Head in Feet.	Pressure in Pounds per Square Inch.		Pressure in Pounds per Square Inch.	Head in Feet.	
	w=62.5	w=62.3		w=62.5	w=62.3
1	0.434	0.433	1	2.304	2.311
2	0.868	0.865	2	4.608	4.623
3	1.302	1.298	3	6.912	6.934
4	1.736	1.731	4	9.216	9.246
5	2.170	2.163	5	11.520	11.557
6	2.604	2.596	6	13.824	13.868
7	3.038	3.028	7	16.128	16.180
8	3.472	3.461	8	18.432	18.491
9	3.906	3.894	9	20.736	20.803
10	4.340	4.326	10	23.040	23.114

If we use the French system the pressure per square centimeter would be 1 gram for every centimeter in height, and 100 grams, or 1 hectogram, for every meter of height or head. Table II gives the weight of distilled water in pounds per cubic foot corresponding to the various degrees of temperature between the freezing and boiling points.

TABLE II. WEIGHT OF DISTILLED WATER.

Temperature (Fahrenheit).	Pounds per Cubic Foot.	Temperature (Fahrenheit).	Pounds per Cubic Foot.	Temperature (Fahrenheit).	Pounds per Cubic Foot.
32°	62.42	95	62.06	160	61.01
35	62.42	100	62.00	165	60.90
39.3	62.424	105	61.93	170	60.80
45	62.42	110	61.86	175	60.69
50	62.41	115	61.79	180	60.59
55	62.39	120	61.72	185	60.48
60	62.37	125	61.64	190	60.36
65	62.34	130	61.55	195	60.25
70	62.30	135	61.47	200	60.14
75	62.26	140	61.39	205	60.02
80	62.22	145	61.30	210	59.89
85	62.17	150	61.20	212	59.84
90	62.12	155	61.11		

The pressure on the small plunger of a hydrostatic press is 45 pounds. Its area is 3 square inches, that of the large plunger is 72 square inches. Find the pressure upon the large plunger.

Solution,—

For each 3 square inches of surface of the large plunger a pressure of 45 pounds is exerted. Hence the total pressure on the large plunger will be $72 \div 3 \times 45 = 1{,}080$ pounds.

EXAMPLES FOR PRACTICE.

1. A hydrostatic press is operated by a man at the long arm of a simple lever which is 4 feet long. The short arm, 3 inches long is attached to the small plunger which has a diameter of 1 inch. If the diameter of the large plunger is 10 inches what pressure will be exerted by the same when the man exerts a force of 40 pounds on the long arm. Ans. 64,000 pounds.

2. What pressure per square foot will be exerted at the bottom of a column of water 48 feet high? Ans. 3,000 pounds.

3. The downward pressure exerted by a column of water is 937.5 pounds per square foot. What is the depth?

Ans. 15 feet.

4. The downward pressure is 145 pounds per square inch. What is the depth? Ans. 334.08 feet.

Average normal pressure against any plane rectangular sur-

face is at $\frac{1}{2}$ of the distance from the surface of the water to the bottom of the rectangular surface. If we were to replace the pressure of the water by a force acting at a single point, that point on a submerged rectangle would be $\frac{1}{3}$ of the way up from the bottom of the rectangle. This point is called the center of pressure.

Let us investigate a case in which water stands at different levels on opposite sides of a plane surface, taking, for example, the cross-section of a self-acting tide-gate, built to drain a salt marsh, as shown in Fig. 17. Let the head of water above the sill S, on the ocean side be h_1 and that on the other side h_2. Let the weight of a cubic foot of water be called w, the pressure on the ocean side P_1 and on the other side P_2. Now the lateral pressure upon any plane surface is equal to the area of that surface multiplied by one-half the head of water acting against that surface. Hence the pressure P_1 on the ocean side is $w \times h_1 \times \frac{1}{2} h_1 = \frac{1}{2} w h_1^2$. We have already seen that the pressure is at a distance of $\frac{1}{3} h_1$ above the bottom of the gate, *i. e.*, the center of pressure is at one third of the head. On the marsh side the head h_2 gives a pressure $P_2 = \frac{1}{2} w h_2^2$ with the center of pressure $\frac{1}{3} h_2$ above the bottom of the gate. Hence the resultant pressure, P, is equal to $P_1 - P_2 = \frac{1}{2} w h_1^2 - \frac{1}{2} w h_2^2$ or,

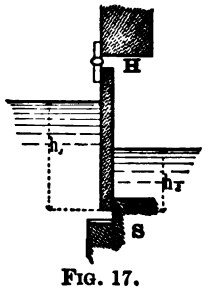


FIG. 17.

$$\frac{1}{2} w (h_1^2 - h_2^2)$$

Suppose we represent the distance above the sill of the point of application by z. Now the moment of P_1 must be $P_1 \times \frac{1}{3} h_1$ and the moment of P_2 must be $P_2 \times \frac{1}{3} h_2$ since the moment of a force is the product of its intensity and the length of its arm. Hence the moment of the resultant pressure must be the difference of the component moments, or

$$(P_1 - P_2)\, z = P_1 \times \frac{1}{3} h_1 - P_2 \times \frac{1}{3} h_2$$

From this equation we may compute z

$$z = \frac{P_1 \times \frac{1}{3} h_1 - P_2 \times \frac{1}{3} h_2}{P_1 - P_2}$$

The action of a gate in resisting water pressure is like that

of a beam under a load, the points of support being the sill and the hinge H. Let the reaction at the hinge be represented by R. Then if h be the height of the gate the reaction at the hinge must be the moment of the resultant pressure divided by h. We may represent this as follows:

$$R = (P_1 - P_2)\frac{z}{h} = \frac{P_1 h_1 - P_2 h_2}{3h}$$

An inspection of Fig. 17 will show that this action has its greatest value when h_1 becomes equal to h and h_2 is zero. If the gate is a vertical one, such as the gate of a canal lock, which swings like a door, a similar problem arises and the above solution holds good.

NOTE. In the above consideration the breadth of the gate is considered as unity, in this case one foot. With a broader gate, the width enters as a factor wherever surfaces are involved.

EXAMPLE FOR PRACTICE.

If the head on one side of a tide-gate is 7 feet and on the other 4 feet, find the resultant pressure and its point of application above the sill. Solution, — The resultant pressure is $\frac{1}{2}\ w\ (h_1^2 - h_2^2)$ then,

$$P = \tfrac{1}{2}\ 62.5\ (7^2 - 4^2)$$
$$= 1031.25 \text{ pounds per linear foot.}$$

Now $(P_1 - P_2)\ z = P_1 \times \frac{1}{3} h_1 - P_2 \times \frac{1}{3} h_2$

$$\text{or } z = \frac{P_1 \times \frac{1}{3} h_1 - P_2 \times \frac{1}{3} h_2}{P_1 - P_2}$$

substituting the pressure and depth values; $z = 2.82$ — feet above the sill.

The pressure on the back of a masonry dam is normal to its surface. If the back is a plane surface the resultant pressure is normal to the plane and the center of pressure is at two-thirds of the water depth. Thus in Fig. 18, AO is two-thirds of AE. Let h be the head of water above the base of the dam, then the normal pressure is as in the previous case, $w \times h \times \frac{1}{2}h = \frac{1}{2}\ wh^2$. From this we see that the total pressure against the dam varies directly as the square of the height.

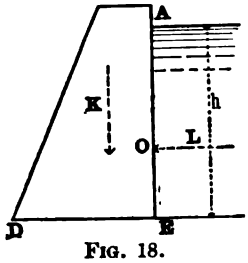


FIG. 18.

It is of importance to understand the amount of pressure against a dam, for the strength of the masonry must be greater than this pressure. There are two cases of failure for dams.

1. The masonry may be pushed by the lateral pressure along the line D E. This happens when the thrust of the head overcomes the resisting force of friction at the base joint.

Let K represent the weight of the masonry, and let f represent the coefficient of friction. Then the resisting friction is $f \times$ K. When the force L, *i. e.*, $\frac{1}{2} wh^2$, is greater than f K the dam slides along the line D E.

2. The second case of failure is by rotation, *i. e.*, the dam may turn about the toe D. This will take place when the moment of L is equal to or greater than the moment of K about the point D. Let us call the lever arms of the forces L and K, l and k respectively. Then the condition for failure by rotation about D is,

$$L\, l = K\, k.$$

Masonry dams are usually so built that the resisting friction is much greater than the horizontal thrust. The common case of failure is, therefore, by rotation and not by sliding.

Let us consider a practical example of these principles. A rectangular masonry dam, (surface vertical on both sides) weighs 140 pounds per cubic foot and is 4 feet thick. What is the head of water upon it which will cause failure by sliding if the coefficient of friction is .75 and what head will cause it to fail by rotation.

We know the horizontal water pressure to be $\frac{1}{2} w h^2$ or $\frac{1}{2}$ $62.5 h^2 = 31.25 h^2$. The resisting friction is k K $= .75 \times 140 \times 4 \times h$. If we place these equations equal to each other and solve for h we shall obtain the height of failure by sliding.

$$31.25 h^2 = .75 \times 140 \times 4 \times h.$$
$$31.25 h = 420.$$
$$h = 13.44 \text{ feet.}$$

To find the height for which failure will occur by rotation, let us consider the weight of the masonry as acting through its center of gravity, *i. e.*, 2 feet from the front lower edge, and the horizontal pressure as acting at $\frac{1}{3}$ h. Then the weight has for its

lever arm 2, and the pressure its lever arm $\frac{1}{3}\ h$. Substituting the values in the equation L l = K k, we get,

$$\tfrac{1}{2} \times 62.5 \times h^2 \times \frac{h}{3} = 140 \times 4 \times h \times 2$$

solving, $h^2 = 107.52$

and, $h = 10.4$ — feet.

From these two values of h we find that the failure occurs from rotation sooner than from sliding.

ORIFICES AND WEIRS.

If water is allowed to escape from a vertical orifice it would, if it were not for the resistance of the air and friction, be thrown to a height equal to the height of the head producing that flow. Another way of stating this would be to say that the velocity of the water escaping from a vessel into a vacuum, the orifice pointing directly upward, would be sufficient to make the water reach the same level as that of the water in the vessel neglecting friction.

It is evident from the law of conservation of energy, that the velocity at the orifice is the same as that acquired by a body falling freely a distance equal to the head of water, or

$$v = \sqrt{2gh} = \sqrt{64.32h} = 8.02\sqrt{h.}$$

By this same formula we see that the head is equal to

$$\frac{v^2}{2g} = .01555v^2.$$

If a vessel or pipe be constantly full of water, all the particles of which move with the same uniform velocity v, and if a is the area of the cross section of the orifice or pipe, the quantity of water which passes any section per second is equal to the volume of a prism whose base is a, (the area of a pipe or orifice) and whose length is v, (the distance traversed by the water in one second). We may embody the above in a formula, as follows:

$$q = av$$

and from this we may also derive the formula that,

$$v = \frac{q}{a}$$

In these formulas q represents the quantity of flow per second. If we designate the total quantity which flows by Q, we have,

$$Q = avt \text{ and } v = \frac{Q}{at}.$$

In these formulas v is in each case equal to $8.02\sqrt{h}$. The head on a vertical orifice is the average head. In the case of a circle this would be the center.

The average head upon a vertical orifice is 16 feet; what is the theoretical velocity of water flowing from that orifice? In the formula $v = 8.02\sqrt{h}$ we may substitute for h its value, then,

$$v = 8.02 \times \sqrt{16}$$
$$= 32.08$$

the required velocity in feet per second. Suppose in the above problem that the area of the orifice is 3 square inches and the time of flow 10 seconds, then the theoretical quantity of flow will be:

$$3 \times 32.08 \times 10 \times 12$$
$$= 11548.8 \text{ cubic inches.}$$

Had the orifice been a circle of 2 inches diameter the theoretical discharge would have been;

$$3.1416 \times 32.08 \times 10 \times 12$$
$$= 12093.9 \text{ cubic inches.}$$

NOTE. 12 occurs as a factor to reduce the velocity from feet per second to inches per second.

EXAMPLES FOR PRACTICE.

1. Find the theoretical velocity of water flowing from an orifice under an average head of 81 feet. Ans. 72.18 feet per second.

2. What is the theoretical discharge from a circular orifice of 4 inches diameter, under a head of 36 feet, flowing for 22 seconds? Ans. 159639.52 + cubic inches.

NOTE. Reduce head to inches.

3. What is the theoretical discharge from a square orifice two inches on a side, if the velocity is 40 feet per second and the time of flow is 30 seconds. Ans. 57,600 cubic inches.

In the measurement of water the standard orifice is used. This term signifies that the opening is so arranged that the water in flowing from it touches only a line. To secure this the inner edge of the opening is a square corner which alone is touched by the water. In precise experiments the orifice may be in a metal plate whose thickness is really small, as may be seen in Fig. 19 at A. More commonly it is cut in a board or plank, care being

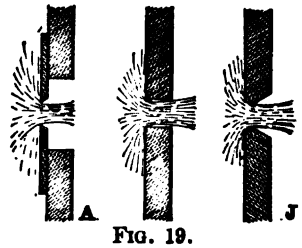


FIG. 19.

taken that the inner edge is a definite corner. It is usual to bevel the outer edges of the orifice as at J, so that the escaping jet shall not touch except at the inner edge. We observe by the figures a certain contraction of the jet, and owing to this contraction the discharge is always less than the theoretic discharge. In order to obtain the actual discharge we multiply the theoretic discharge by a coefficient c, *i. e.*, in the formula $Q = avt$, we place another factor called the coefficient of discharge c. Thus our formula becomes: $$Q = cavt,$$
where c is the coefficient of discharge, a the area of the orifice, v the velocity (theoretical) and t the time. Our units would ordinarily be square feet, feet per second, and seconds. The result would be a total discharge in cubic feet. The following tables give the coefficients for circular, square, and rectangular (one foot wide) vertical orifices.

Table III. Coefficients for Circular Vertical Orifices.

Head H in Feet.	Diameter of Orifice in Feet.						
	0.02	0.04	0.07	0.1	0.2	0.6	1.0
0.4		0.637	0.624	0.618			
0.6	0.655	0.630	0.618	.613	0.601	0.593	
0.8	.648	.626	.615	.610	.601	.594	0.590
1.0	.644	.623	.612	.608	.600	595	.591
1.5	.637	.618	.608	.605	.600	.596	.593
2.	.632	.614	.607	.604	.599	.597	.595
2.5	.629	.612	.605	.603	.599	.598	.596
3.	.627	.611	.604	.603	.599	.598	.597
4.	.623	.609	.603	.602	.599	.597	.596
6.	.618	.607	.602	.600	.598	.597	.596
8.	.614	.605	.601	.600	.598	.596	.596
10.	.611	.603	.599	.598	.597	.596	.595
20.	.601	.599	.597	.596	.596	.596	.594
50.	.596	.595	.594	.594	.594	.594	.593
100.	.593	.592	.592	.592	.592	.592	.592

This table shows that the coefficient of discharge decreases as the size of the orifice increases, and that for diameters less than 0.2 feet it decreases as the head increases. The coefficients above the horizontal lines in the last three columns may not be used in actual problems where the actual discharge is to be computed. The error in these coefficients depends upon the ratio of the diameter of the orifice to the head.

Table IV. Coefficients for Square Vertical Orifices.

Head H in Feet.	Side of the Square in Feet.						
	0.02	0.04	0.07	0.1	0.2	0.6	1.
0.4		0.643	0.628	0.621			
0.6	0.660	.636	.623	.617	0.605	.598	
0.8	.652	.631	.620	.615	.605	.600	.597
1.0	.648	.628	.618	.613	.605	.601	.599
1.5	.641	.622	.614	.610	.605	.602	.601
2.	.637	.619	.612	.608	.605	.604	.602
2.5	.634	.617	.610	.607	.605	.604	.602
3.	.632	.616	.609	.607	.605	.604	.603
4.	.628	.614	.608	.606	.605	.603	.602
6.	.623	.612	.607	.605	.604	.603	.602
8.	.619	.610	.606	.605	.604	.603	.602
10.	.616	.608	.605	.604	.603	.602	.601
20.	.606	.604	.602	.602	.602	.601	.600
50.	.602	.601	.601	.600	.600	.599	.599
100.	.599	.598	.598	.598	.598	.598	.598

This table shows similar variations to Table IV. If the head is less than 4 times the side of the orifice an error is committed in finding the coefficient by the same method as in finding it for a greater head. The coefficients above the horizontal lines are not to be used in actual problems.

Table V. Coefficients for Rectangular Orifices 1 Foot Wide.

Head H in Feet.	Depth of Orifice in Feet.						
	0.125	0.25	0.50	0.75	1.0	1.5	2.0
0.4	0.634	0.633	0.622				
0.6	.633	.633	.619	0.614			
0.8	.633	.633	.618	.612	0.608		
1.0	.632	.632	.618	.612	.606	.626	
1.5	.630	.631	.618	.611	.605	.626	.628
2.0	.629	.630	.617	.611	.605	.624	.630
2.5	.628	.628	.616	.611	.605	.616	.627
3.0	.627	.627	.615	.610	.605	.614	.619
4.0	.624	.624	.614	.609	.605	.612	.616
6.0	.615	.615	.609	.604	.602	.606	.610
8.0	.609	.607	.603	.602	.601	.602	.604
10.0	.606	.603	.601	.601	.601	.601	.602
20.0				.601	.601	.601	.602

From this table we see that the coefficient of discharge for a rectangle of constant breadth increases as its depth decreases. In selecting a coefficient for use with an orifice whose size falls without the limits of the table, it should be borne in mind that large orifices have a smaller value for the coefficient than small orifices.

One way of measuring is by the use of the miner's inch which may be roughly defined as the quantity of water which will flow from a vertical standard orifice 1 inch square if the head on the center of the orifice is $6\frac{1}{2}$ inches. From the fourth table the coefficient of discharge is seen to be about .623. Hence the actual discharge in cubic feet per second would be

$$Q = \frac{.623 \times 8.02}{144} \sqrt{\frac{6.5}{12}} = .0255 \text{ cubic feet.}$$

The discharge in 1 minute would be $60 \times .0255 = 1.53$ cubic feet. The mean value of the miner's inch is therefore roughly about 1.5 cubic feet per minute. It differs considerably, however. When water is bought for mining or irrigating purposes, a much larger quantity than the miner's inch is required and various methods of measurement are used.

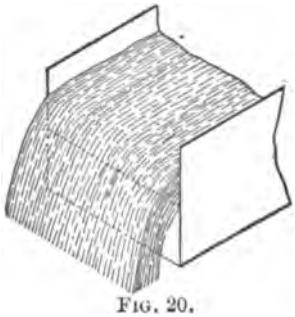


FIG. 20.

A weir is a notch in the top of the vertical side of a reservoir through which water flows. The notch is usually rectangular, and is shown in the following illustration, Fig. 20. In measurements of water flowing over weirs the same precaution should be taken as for orifices, namely, that the inner edge shall have a definite angular corner. The theoretic discharge of a rectangular weir is found to be,

$$q = \tfrac{2}{3} \sqrt{2g}\, b\, H \sqrt{H}.$$

In this formula b is the breadth of the notch, commonly called the length of the weir, g the acceleration of gravity, and H the depth of water on the lower edge. This depth must not be measured in the plane of the weir, but the height must be taken a considerable distance back from the weir on account of the decrease in head where the water flows over; H is ordinarily determined by the head a certain distance back from the weir, and determining

an extra head to be added to this by ascertaining the velocity with which the water approaches the weir. In order to get the actual discharge, a coefficient of discharge must be used, and its computation is rather beyond the province of this work.

If water flows through a pipe the diameter of which varies, the theoretical velocity varies inversely as the square of the diameter of the pipe, because the same quantity passes through each section of the pipe in the same time, but in some places the area is greater, therefore the velocity need not be so great. Friction causes a loss of effective head as we go along the pipe. The longer the pipe, as we know, the more slowly water flows. If small tubes were inserted along the pipe, the water would rise in them, but as we approached the end of the pipe from which the water flows the heights of the water in the tubes would decrease. The head acting to cause flow throughout the remainder of the pipe decreases as we go towards the end from which the water flows.

Let us find the probable discharge in 10 seconds from a vertical orifice, .2 feet in diameter under a head of 10 feet. The velocity is 25.4 feet per second from the formula $v = 8.02 \sqrt{h}$. The time t, is ten seconds, the area a of the orifice is .031416 square feet, and the coefficient of discharge c from Table III. is .597. If we substitute these values in the formula $Q = c\ a\ v\ t$, our equation becomes,

$$Q = .597 \times .031416 \times 25.4 \times 10$$
$$= 4.76 + \text{cubic feet.}$$

EXAMPLES FOR PRACTICE.

1. Find the probable discharge in 5 seconds from a circular vertical orifice .1 foot in diameter if the head is 20 feet.

Ans. .839 + cubic feet.

2. The theoretical discharge from a square, vertical orifice is 4.96 cubic feet per minute. If the coefficient of discharge is .602 what will be the probable amount of flow in 15 seconds?

Ans. .746 + cubic feet.

3. What is the probable discharge from a rectangular orifice one foot wide, and .75 feet deep if the theoretical flow for 10 seconds is 12.03 cubic feet and the head is 4 feet?

Ans. 7.3 + cubic feet.

4. Express 22.5 cubic feet per minute in miner's inches whose value is 1.5 cubic feet per minute. Ans. 15.

5. If a pipe supplies 900 cubic feet of water per minute, what is the rate of flow expressed in miner's inches of the same value as above? Ans. 600.

6. What is the theoretic discharge of a waste weir whose length (b) is 2 feet if the depth of water on the lower edge is 3 inches?

NOTE.—Substitute in the equation $q = \frac{2}{3}\sqrt{2g}.\ b\ H\sqrt{H}$.

Ans. 1.337— cubic feet per second.

WATER WHEELS.

Water wheels are often used to transfer the energy of streams. They are of different kinds, the availability of any one being determined by the nature of the water supply and the work to be done.

One form of water wheel, namely, the overshot, Fig. 21, depends for its energy upon the amount of fall which a stream has, rather than upon the amount of water it can furnish. The water falls into buckets at the top and the weight of water together with the current forces the wheel to turn. The buckets become inverted as the wheel turns and when at the lowest part of their path are very nearly empty. Such wheels have been made nearly 100 feet in diameter and utilize as high as 80 per cent of the stream's energy.

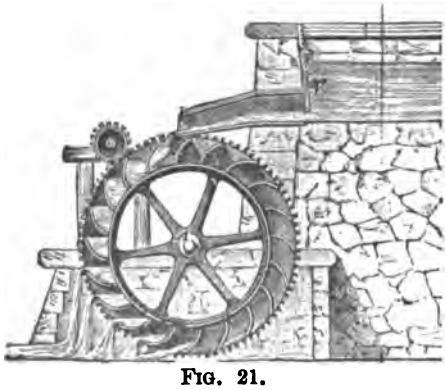


FIG. 21.

A breast wheel, Fig. 22, receives the water, as the name implies, at its breast. Float boards are fixed perpendicular to the circumference and as the stream is received at or near the level

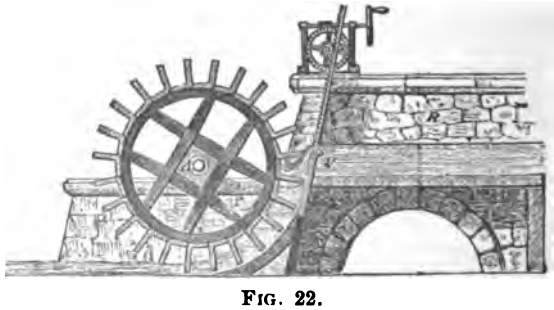


FIG. 22.

of the axis both the weight of water and the force of current may be utilized. This wheel is not as efficient as the overshot and depends for its energy to a large extent upon the volume of water.

The undershot wheel, Fig. 23, may have a similar construction to the breast wheel. The stream strikes a few float boards near the bottom of the wheel. These boards are more or less submerged and the wheel is driven therefore by the force of the current. This form is the least efficient of those given. It requires a large amount of swiftly moving water if much power is desired. The amount of energy which it can furnish is in many cases not more than 25 per cent of the stream's energy.

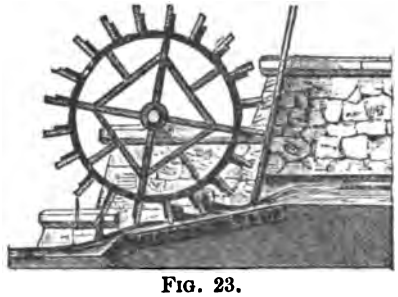


FIG. 23.

The turbine is the most effective of the many varieties of wheel. Its efficiency often is as high as 85 per cent of the energy

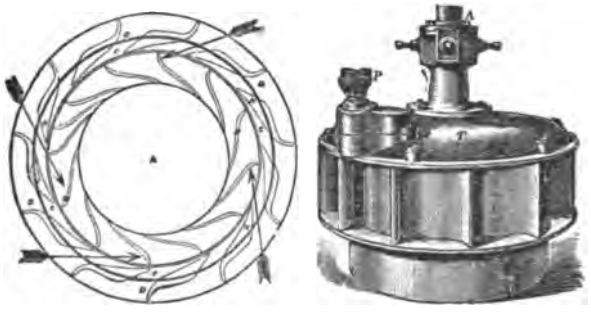


FIG. 24.

furnished. Let us examine the construction and action of this wheel. A type of turbine known as the inward flow wheel, is shown in Fig. 24, which gives a perspective and also a horizontal section through the wheel and case. It must be kept wholly submerged in water, otherwise it would be merely an impulse wheel. The wheel itself, *i. e.* the part which revolves, is represented by the letter B. This part and its enclosing case D are placed in a penstock under a considerable head of water. The arrows show the direction in which the water enters through the openings in D. These openings are so constructed that when the water

strikes the buckets of B the greatest force of the water is utilized in moving B. The wheel B moves with the greatest velocity where for a given head we change the direction of flow of water to the greatest extent. Power is conveyed by the central shaft A to any machinery desired. The water after leaving the buckets escapes by a vertical draft tube below the turbine. The wheel itself rests upon a central pivot which is carried by cross-arms from the bottom of the outer case. This case in turn is covered by a top T which relieves the wheel from any vertical pressure that the head of water may exert. The water passes to the wheel through openings called chutes. It is often necessary to govern the amount of water used. For this purpose a cylindrical collar, C, is placed between the wheel and the outer case. The collar or register gate may be moved by the pinion P and thus opens or closes the chutes. Care should be taken that the water passages are not so large that the velocity of the water through them shall be more than one and a half feet per second. For with a higher velocity the greatest force of the water is not obtained. The great disadvantage of the turbine is that we cannot decrease its water supply without materially decreasing its efficiency. If therefore the water supply of a given plant should partly fail, the turbines if not given the amount they need, will not use the available supply with a high efficiency.

PNEUMATICS.

Pneumatics is the branch of physics which treats of aeriform bodies, their mechanical properties, and the machines in which they are utilized. However small the quantity of a gas, it always fills the vessel in which it is held. A bladder, or India rubber bag, partially filled with air, and having the opening tightly closed, is placed under the receiver of an air pump and the air exhausted from the receiver, or jar in which the bladder is placed. The bladder immediately swells to a large size, because the air is removed from the jar until its pressure on the outside of the bag is less than the pressure of the air against the inner surface of the bag. As a result, the air in the bag expands to take the place of the air which has been withdrawn from the jar. When dealing with gases the word tension is used to designate the force with which they resist pressure.

As water was taken as a type of liquids, so we may take atmospheric air as the type of aeriform bodies, and the properties in general which belong to air may be understood as belonging to all gases.

Air is chiefly a mixture of two gases, oxygen and nitrogen, with the proportions of about 1 to 4 by volume. This atmosphere surrounds the earth and exerts a certain pressure on its surface, just as the sea water exerts pressure, due to its weight, upon the bottom of the ocean. This pressure exerted by the air is about 14.7 pounds per square inch. It varies considerably, as we shall see later.

We may easily ascertain the fact that air has weight, by pumping as much air as we can out of a jar. We then find that it weighs less than it did before air was pumped out. We cannot remove all the air. If we have a U-shaped tube, Fig. 25, filled with mercury, the mercury stands at equal heights in each arm. If, however, we fill one arm, then stop it up with a cork, as shown at a′ in the part of the figure marked B, and hold the tube in an upright position, we see that we do not have equal heights in both arms. In the first place, in the part of the figure marked A, we have the arm ab balancing the arm cd.

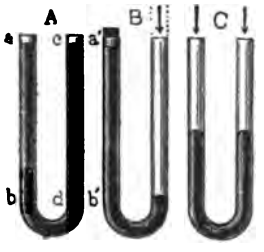


FIG. 25.

Now in the case of B, what supports the arm a′b′? It is the pressure of the atmosphere acting downward in the tube in the direction shown by the arrow. If we remove the cork at a′ the atmosphere immediately presses upon the mercury with the same force that it does in the other arm, and the mercury immediately takes the same level in each of the arms, the pressure of the atmosphere on each arm being the same.

FIG. 26.

If now we take a U tube, Fig. 26, which is about 3 feet long, and fill one of the arms with mercury, stopper it, and hold the tube in an upright position, we find that the mercury no longer completely fills the arm, but is distant from the closed end of the tube by a distance ac. This is because the air does not exert pressure enough to counter-

balance as great a height of mercury as a column equal to ab in length. It can only support a column of mercury equal to cb. This length is usually about 30 inches.

FIG. 27.

This length of column supported by the atmosphere varies according as the pressure of the atmosphere varies. Take Fig. 27. The air presses upon the surface of the mercury in the vessel marked nn, with a certain pressure. The mercury in the column must press upon the surface of the mercury in the vessel with just the same pressure. If the pressure of the air is 15 pounds per square inch, the pressure of the mercury in the tube is 15 pounds per square inch. The greater the pressure of the air the higher the mercury column.

This arrangement is called a barometer. It consists of a closed tube of mercury inverted and arranged so that the pressure of the atmosphere may act upon the mercury. Such an arrangement is shown in Fig. 28. As the pressure of the air column is equal to the difference in height between the long and short arms, the short arm is made in a bulb form so that the position of its surface will change but slightly for a considerable change in the long arm.

FIG. 28.

A form of barometer that is very convenient to use is the aneroid, Fig. 29. It occupies very little space, is easily portable, and at the same time shows very slight variations of atmospheric pressure. It consists of a cylindrical metal box, with a back of some thin, corrugated, elastic metal. The face is provided with a dial graduated by a scale which is referred to the standard mercurial barometer. When the air is removed from the box any increase

of atmospheric pressure acts on the back to force it inward. Whenever the atmospheric pressure decreases, the back moves outward by its own elasticity and assisted by a spring. The motion of this metal back is communicated by means of delicate multiplying levers to an index at the face. The index in turn moves over the dial thus indicating every variation of atmospheric pressure. This barometer requires careful usage and must often be referred to the standard mercury barometer, otherwise its readings cannot be depended upon.

FIG. 29.

EXAMPLES FOR PRACTICE.

1. When the height of the mercury column of the barometer (Specific Gravity of Mercury 13.6) is 76 centimeters; how tall a column of water can the pressure of the atmosphere support in a tube, there being a perfect vacuum at the top?

Ans. 10.336 meters.

2. What air pressure is exerted upon a surface 8 inches long and 9 inches wide, if the atmospheric pressure is 14.7 pounds per square inch? Ans. 1058.4 pounds.

3. If the atmospheric pressure is 14.9 pounds per square inch, what is the height of the mercury column in the barometer, if the Specific Gravity of mercury is 13.62?

Ans. 30.2+ inches.

SUGGESTION. Calculate the height of a water column that would give a pressure of 14.9, and divide by the specific gravity of mercury.

4. When the barometer stands at 764 mm, what is the atmospheric pressure per square centimeter of surface, considering the Specific Gravity of mercury to be 13.6? Ans. 1039.04 grams.

MARIOTTE'S LAW.

The following law, first formulated by Mariotte, is of primary importance in dealing with gases. It is as follows:

The temperature remaining the same, the volume of a given quantity of gas is inversely proportional to the pressure.

This means that in order to diminish the volume of a gas one-half, the pressure must be doubled; to diminish it to one-third, the pressure must be three times as great. A pressure four times as great would reduce it to one-fourth of its original volume.

In dealing with applications of this law, we must always have the absolute pressures, *i. e.*, we must reckon the atmospheric pressure which we may find from the barometer. Suppose, for example, 4 cubic feet of air to be under a pressure of 60 pounds per square inch in a cylinder fitted with a movable piston. If now the pressure is removed, the piston moves out allowing the gas to take up greater space, and we will suppose that the pressure on the gas in the cylinder has decreased to an atmospheric pressure of 15 pounds. The volume is then 4 times as great as it was before, or 16 cubic feet. We see that in each case the product of the volume and pressure is the same. In the first place we have,

$$4 \times 60 = 240$$

and in the second case,

$$16 \times 15 = 240$$

In other words, the product of the pressure and the corresponding volume for any position of the piston is the same as it is for any other position.

In order to embody the above work in formulas, let P equal the original pressure, p the final pressure, V the volume corresponding to the pressure P, and v the volume corresponding to the pressure p. Then

$$p = \frac{PV}{v}.$$

Suppose a certain quantity of air measures one liter. Under what condition can it be made to occupy 500 cu. cm.? To find the pressure necessary we divide the original by the final volume Or, substituting in the above equation,

$$p = \frac{1 \times 1000}{500} = 2$$

Hence the pressure necessary is twice the original.

The formula just given may be written also in the following manner.

$$v = \frac{PV}{p}$$

EXAMPLES FOR PRACTICE.

1. What is the final volume of 4 liters of steam under the ordinary atmospheric pressure of 14.7 pounds per square inch, if the pressure is increased to 29.4 pounds per square inch.

Ans. 2 liters.

2. The final pressure is one-eighth of the original. If the original volume is 2,000 cubic inches, what is the final volume?

Ans. 16,000 cubic inches.

3. Under what conditions can a liter of hydrogen be made to occupy 2,000 cu. cm.? Ans. One-half the original pressure.

4. The pressure upon a cylinder of steam is 75 pounds per square inch. If the volume of the cylinder is 15,552 cubic inches, and the pressure increased until the steam occupies 6,912 cubic inches, what is the final pressure?

Ans. $168\frac{3}{4}$ pounds per square inch.

The foregoing formulas may be embodied in words as follows:

To obtain the final pressure, divide the product of the original volume and pressure by the final volume.

To obtain the final volume, divide the product of the original volume and pressure by the final pressure.

The pressure exerted by a gas in opposition to a pressure acting upon it is called its tension, and the density of a gas is inversely proportional to its volume, because the greater the volume for a given weight the less the density. We may sum up all our results by the following equation.

$$\frac{V}{v} = \frac{p}{P} = \frac{d}{D}$$

D represents the original density, and d the final density.

The weight of twenty cubic feet of a gas is 1 pound. After

compressing to $\frac{1}{2}$ its original volume, what will be the weight per cubic foot? From the above formula the density of a gas is proportional to its volume, or

$$\frac{V}{v} = \frac{d}{D}$$

Substituting the values given this becomes,—

$$\frac{20}{10} = \frac{x}{1} \quad \text{or} \quad \begin{aligned} 10\,x &= 20 \\ x &= 2 \end{aligned}$$

Hence we see that the density after compression is twice the original. Now if 10 cubic feet after compression weighs 1 pound the weight per pound must be $1 \div 10$ or .1 pound per cubic foot.

PUMPS.

Atmospheric pressure is utilized in the lifting or suction pump. If a partial vacuum be formed at the surface of any liquid the pressure upon the remaining surface is an unbalanced force and the liquid is forced into the space thus formed. Experiments have shown that the atmospheric pressure will (theoretically) support a column of water 34 feet high. Let us see how we may apply this principle in order to lift water. Fig. 30 shows a cross section of a suction pump. It consists of a cylinder or barrel, piston, valves, and a suction pipe which dips below the surface of the water. The piston with its two valves fits tightly to the walls of the cylinder. These valves and the one leading from the suction pipe to the cylinder open only in one direction, *i. e.*, upward.

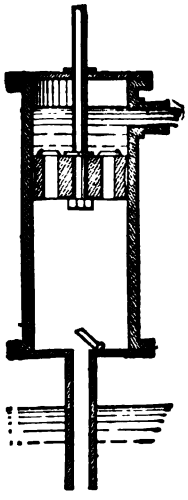


FIG. 30.

Suppose the piston to be at the bottom of the cylinder and all valves closed. Now raise the piston and we find that the air pressure upon the piston valves keeps them closed, at the same time a partial vacuum is created below the piston. Then the water rushes up the suction pipe through its valve and fills the lower part of the cylinder. When the piston is lowered the suction pipe valve closes immediately, and the water can find no escape except through the piston

valves. The force of compression exerted by the piston forces the water through its valves. Then the piston is again raised, and this body of water above the piston is forced from the cylinder through the outlet, since it cannot go back through the piston. At the same time the piston tends to create a vacuum below itself as before, and the water rises to destroy that action.

The reservoir containing the water to be pumped must not be contained in a tightly-closed vessel, since the atmospheric pressure must act upon the surface of the liquid. Owing to mechanical imperfections the practical limit of the suction is 28 feet vertically from the barrel to the surface of the water. If we wish to raise the water higher, we may use a longer piston rod and thus place the barrel nearer the surface of the water. Then by means of a long pipe from the top of the barrel the water may be raised to a considerable height.

It is customary to use a force pump when the water is to be pumped to any considerable height. Fig. 31 shows a cross section of the pump and its air-chamber with the piston partly raised. The upper valve is placed at the inlet of the air-chamber. The suction pipe is placed as in the previous case, Fig. 30. When the piston is raised water is forced up into the barrel by atmospheric pressure. When the piston is forced down the suction valve closes and the water is forced through the other valve into the air-chamber. As soon as the piston is again raised the pressure in the air-chamber closes the valve at its inlet, at the same time more water is forced from below into the barrel.

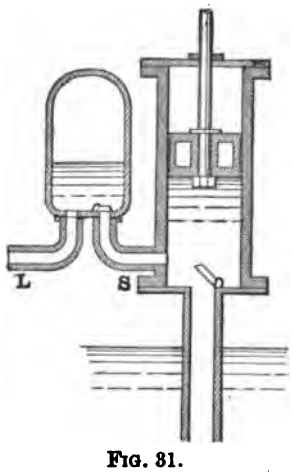


FIG. 31.

It will be noticed that such a pump without its air-chamber does not furnish a steady supply of water. This difficulty is overcome by connecting the discharge pipe S with the air-chamber. Both entrance and exit must be below the surface of the water, otherwise the air would be forced from the chamber. When water is forced into the air-chamber the mouth of the delivery pipe L is

covered and the air compressed. As soon as the tension of the air is great enough water is forced from the delivery pipe. Now the air acts as a sort of cushion. It absorbs the intermittent force which the pump exerts in driving the water, but gives it out again as a constantly acting force. Therefore the water is driven from the outlet L in a constant stream.

SECTION 6.

HEAT.

Heat is a form of energy which consists of vibratory motions of the molecules of matter, or results from these motions. It gives rise to the sensations of warmth and cold. As it is a form of energy it is measurable though not a material substance.

The temperature of a body is its state with reference to its ability to communicate heat to other bodies. Temperature merely shows how hot a body is, and not the amount of heat a body possesses. Cold is simply a comparative term signifying a lesser degree of heat. When a body receives heat its temperature generally rises, but at a certain point a change of condition sometimes results instead. If a body gives up heat, its temperature falls, or else its physical condition changes. A person cannot judge as to whether a body is warm or cold by simply touching it, for this depends on the condition of the person.

If the hand were very warm and were placed in water of ordinary temperature, the water would be thought cold; but if the hand were cold and placed in water at ordinary temperature, the same water would seem warm. If one hand is placed in very cold water and the other hand in very hot, and then both hands placed in water at ordinary temperature the water would seem warm to the hand which had been in the cold water, and would seem cold to the hand which had been in the hot water.

We determine temperature by means of an instrument called the thermometer, the most common form being the mercury thermometer which consists of a glass tube containing mercury. When the mercury is heated it expands and nearly fills the tube; when cold it contracts. We graduate the thermometer according to two fixed points, the freezing and the boiling points of water, *i.e.*, the temperature of melting ice and the temperature of steam as it escapes from water boiling under a pressure of one atmosphere.

The thermometer of the English system is the Fahrenheit, Fig. 1. The freezing point of this is marked 32 degrees, and

the boiling point 212 degrees. In the French system the Centigrade thermometer is used. The freezing point on this is marked 0, and the boiling point 100. Another thermometer is the Reaumur. The freezing point on this is marked 0 and the boiling point 80.

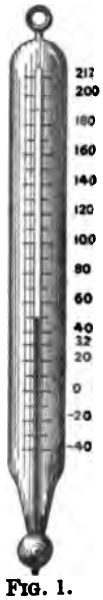


FIG. 1.

If we wish to reduce a Fahrenheit reading to a Centigrade reading we subtract 32 and divide by 1.8. We do this because the Fahrenheit freezing point is 32 degrees above 0, and as the distance between the freezing point and boiling point of the Centigrade is 100 degrees and of the Fahrenheit 180 degrees, it takes 1.8 degrees Fahrenheit to equal 1 degree Centigrade.

To change a Centigrade reading to a Fahrenheit reading, we multiply by 1.8 and add 32. We may represent these methods of conversion by the following formulas, where C. equals the degree Centigrade, and F. the degrees Fahrenheit.

$$C. = \tfrac{5}{9}\ (F. - 32)$$

$$F. = \tfrac{9}{5}\ C. + 32$$

NOTE. $\frac{5}{9}$, since $1.8 = \frac{9}{5}$.

EXAMPLES FOR PRACTICE.

1. How many degrees F. would a temperature of 68° C. indicate? Ans. $154\frac{2}{5}$° F.

2. A Fahrenheit thermometer reads 167°. What is this temperature expressed in Centigrade degrees? Ans. 75° C.

3. The difference between the temperatures of two bodies is 36° F. Express the difference in Centigrade degrees. Ans. 20° C.

4. The sum of the temperatures of two bodies is 35° C. Express this sum in Fahrenheit degrees. Ans. 63° F.

EXPANSION.

As we have seen, heat is a form of energy consisting of molecular vibrations. If the temperature of a body is increasing, the energy with which its molecules fly back and forth, striking each other, becomes greater. Of course these molecules are far too

small and their range of motion too infinitely minute to permit of observation so we are obliged to imagine what has taken place. These molecules are held together by the force of cohesion, yet they are vibrating rapidly. The higher the temperature the greater the energy of these molecular vibrations. This vibrating of the molecules causes them to occupy a greater amount of space. They push themselves farther apart and thus cause the body, of which they form a part, to take up more room, *i. e.*, to expand. This expansion or increase of volume is the first effect of an increase in heat. If a bar, say of iron, is increased in temperature we find that it lengthens. If we have an iron ball which will just pass through a ring, and we heat the ball we find that it has so increased in size that it will no longer pass through the ring. If we allow it to cool, it will return to its former size so that it can pass through. This is an illustration of the increase of volume or cubical expansion. Often we measure simply the increase in length because it is with this that we commonly deal. This expansion is termed linear expansion.

If we have bars of two substances, such as iron and brass, and both are heated equally, we find that one is lengthened more than the other; the brass more than the iron. If a thin plate of brass is fastened to one of iron by riveting and the two heated, we find that the bar not only lengthens but becomes curved in shape instead of remaining straight, the brass being on the convex side. This is because the brass has expanded more than the iron, and so become longer.

In most cases liquids also expand. Suppose we take water at ordinary temperature and heat it. It increases in size, *i. e.*, it decreases in density. This is true for any increase in temperature above 4° C., or 39.2°F. If, however, we decrease its temperature below the above-named point, instead of decreasing in size and increasing in density, it increases in size and decreases in density and when it reaches the freezing point, 0° C. or 32°F. it changes to a solid, ice, the change however, entailing a considerable loss of heat. The ice thus formed is much less dense than the water and takes up more room, *i. e.*, its volume is greater. We can readily see that the density is less because ice floats upon water. If water freezes inside a closed vessel and the vessel is unable to stretch

sufficiently, it is broken. Ice does considerable work geologically by freezing in the fissures of rocks and thus splitting off fragments. This fact is utilized in the production of sheets of slate.

Another instance of water becoming lighter after dropping below the temperature of 4°C. is found in the water of lakes. In even the coldest winters cold water floats upon the warmer which is more dense and the ice forms at the surface thus protecting the life beneath.

When expansion takes place, it is found that the increase of size is proportional to the size of the body and to the number of degrees through which it is heated. It is also dependent upon the nature of the body. This relation is expressed by means of a constant called the coefficient of expansion of the body. Suppose for instance that a rod 10 feet in length at 0° C. is heated to 100° C. and its length is found to be 10 feet 1 inch. The difference of length, namely 1 inch, is evidently the linear expansion. Let us call the coefficient of expansion c, the linear expansion e, the number of degrees the body has been heated t, and the length of the body before heating l. From what has been stated we may write the equation:

$$e = c\,t\,l.$$

Solving for c we get:

$$c = \frac{e}{t\,l}$$

Substituting the above values our equation becomes:

$$c = \frac{1}{10 \times 12 \times 100} \text{ or,}$$

$$c = .000083 +$$

Hence we may define the coefficient of linear expansion as the amount of linear expansion which a unit length of a body undergoes for one degree change in its temperature. The volume coefficient is similarly defined, volumetric being substituted for linear and volume for length.

Another definition of the coefficient of expansion is that it is the ratio of the expansion per degree rise in temperature to the length of the body (the length being determined for 0° C. or 32° F.)

The following table gives values in accord with the best authority, though differences of temperature and material will affect the results. The substances were experimented upon at ordinary temperatures.

COEFFICIENTS OF LINEAR EXPANSION (t, Ordinary.)

Substance.	For 1°Fahrenheit.	For 1°Centigrade.
	Length = 1.	Length = 1.
Aluminum	.00001234	.00002221
Brass (cast)	.00000957	.00001722
Copper	.00000887	.00001596
Glass (flint)	.00000451	.00000812
Iron (wrought)	.00000648	.00001166
Iron (cast)	.00000556	.00001001
Lead	.00001571	.00002828
Masonry, brick	.00000375	.00000675
Mercury	.00003328	.00005990
Platinum	.00000479	.00000863
Silver (pure)	.00001079	.00001943
Steel (cast)	.00000636	.00001144
Steel (tempered)	.00000689	.00001240
Tin	.00001163	.00002094
Wood (pine)	.00000276	.00000496
Zinc	.00001407	.00002532

Cubical expansion coefficients, or coefficients of expansion of volume may be found approximately by multiplying the coefficient of linear expansion by 3.

Let us now determine the length of a substance after expansion. We may denote the length of the substance at 0°, l_o, the given length by l_t, and the required length by l_r. As we have seen:

$$e = c\,t\,l_o, \text{ and}$$

evidently $l_t - l_o = e$, then substituting the value of e in this second equation it becomes:

$$l_t - l_o = c\,t\,l_o$$

solving for l_t we get:

$$l_t = l_o + c\,t\,l_o$$

$$\text{or, } l_t = l_o\,(1 + c\,t) \qquad (1.)$$

solving for l_o we get:

$$l_o = \frac{l_t}{1 + c\,t}. \qquad (2.)$$

From equations (1.) and (2.) we see that to obtain the length of a solid at any temperature, having its length at zero we have only to multiply its length at zero by $(1 + c\ t)$. To obtain the length at zero we simply divide the given length by $(1 + c\ t)$.

Suppose the length is given at a certain temperature $t°$, and it is required to find the length l_r at a temperature t_r. An inspection of equation (1.) will show that $l_r = l_0\ (1 + c\ t_r)$. Dividing this equation by (2.) we obtain:

$$\frac{l_r}{l_t} = \frac{1 + c\ t_r}{1 + c\ t} \quad \text{or,}$$

$$l_r = l_t \frac{1 + c\ t_r}{1 + c\ t}$$

This formula may be used to determine the length of any bar which has been heated through known temperatures, when the previous length and the coefficient of expansion are known. A more convenient form for this formula which will give approximate results is:

$$l_r = l_t\ [1 + c\ (t_r - t)\ ]$$

How may we find the length of a bar of wrought iron 40 feet long which has been heated from 60° to 110° C.? We may use the above formula which becomes if the proper values are substituted:

$$l_r = 40\ \Big(1 + .00001166\ (110 - 60)\Big)$$
$$= 40.02332$$

Hence the length of the bar after expansion is 40.02332 feet. This result is probably correct to less than hundredths of an inch. It will be noticed that the value of c has been taken from the table of coefficients of expansion, and the student should use these values in preference to computing by the method previously given, as they are more accurate than most observers can find.

EXAMPLES FOR PRACTICE.

1. A bar of cast steel, 200 inches in length, is heated from 40° to 160°F. What is its length at the latter temperature?

Ans. 200.15264 inches.

2. The length of a sheet of copper is 10 feet, and its temperature is 25°C. Suppose it be heated to 95°C., what will be its length? Ans. 10.011172 feet.

EXPANSION OF GASES.

If we partially fill a bladder with cold air and then place it near the fire the bladder becomes distended by the same air which was originally contained in but did not fill it, in other words the air has expanded. The rate of this expansion is practically the same for all gases and is $\frac{1}{273}$ or .00366 of the volume at freezing for each degree C. rise of temperature and $\frac{1}{490}$ for each degree F. above 32 degrees. That is to say if a gas at 0° C. is raised 1° in temperature it increases $\frac{1}{273}$ of its volume, or it increases from $\frac{273}{273}$ to $\frac{274}{273}$. If we raise the temperature 2° C. the volume of the gas increases $\frac{2}{273}$ of what it was at 0°. Hence it becomes $\frac{275}{273}$ of what it was at 0°.

If we have a liter of gas at 0° C. and increase its temperature to 10° C. its volume becomes $1 + \frac{10}{273}$ liters. We may embody this in a formula as follows, where n equals the number of degrees increase in temperature.

$$\text{Volume at } t \text{ degrees C.} = (1 + \frac{n}{273}) \times \text{the volume at } 0° \text{ C.}$$

To write as a decimal,

$$\text{Volume at } t \text{ degrees C.} = (1 + .00366n) \times \text{volume at } 0° \text{ C.}$$

If we use degrees F. our expansion, as we know, will only be $\frac{5}{9}$ as great per degree, because 1° C. = 1.8° F. Thus our formula becomes for degrees F. where n = number of degrees above freezing point.

$$\text{Volume at } t \text{ degrees F.} = (1 + \frac{n}{490}) \times \text{the volume at } 32° \text{ F.}$$

If the temperature is less than 32° we subtract the second quantity in the parenthesis instead of adding it.

Let us find the volume at 150° C. of a gas that measures 10 cubic cm. at 15° C. First we must find the volume at 0° C, which may be obtained by substituting the proper values in the above equation. Then, if x is the volume at 0° C.,

$$10 = (1 + \frac{15}{273})\, x = \frac{273 + 15}{273}\, x = \frac{288}{273}\, x,$$

$$\text{or } x = \frac{273 \times 10}{288}.$$

Now the volume at 150° C. may be found directly from this same equation, if we use 150 as the value of n. Then, the required volume $= (1 + \frac{150}{273})\ (\frac{273 \times 10}{288}) = 14.69$ cubic centimeters (nearly).

EXAMPLES FOR PRACTICE.

1. What will be the volume of 900 cubic cm. of oxygen at 0° C, when heated to 30° C.? Ans. 998.9 + cubic centimeters.

2. A certain weight of air measures a liter at 0° C. How much will the air expand on being heated to 100° C.?

Ans. 366.3 + cubic centimeters.

3. If 100 cu. cm. of hydrogen be measured at 100° C., what will be the volume of the gas at — 100° C.?

Ans. 46.38 + cubic centimeters.

4. If 170 cu. dm. of carbonic acid gas be measured at 10° C, what will be the volume when the temperature sinks to 0° C.?

Ans. 164 cubic decimeters (nearly.)

LIQUEFACTION.

As we have seen, cohesion holds the molecules of a substance together, while heat tends to keep them in general farther apart, and the condition of the substance depends upon which one of the two predominates. If cohesion predominates we have the solid form, while if there is a sufficient amount of heat, cohesion is overcome and the solid becomes a liquid, or it melts.

If we heat a substance until it melts the temperature at the beginning of melting remains constant until the melting is finished, *i. e.*, from the beginning to the end of melting the temperature is the same. Melting is also called fusion. The laws of fusion are as follows:

1. *Every solid begins to melt at a certain temperature and this temperature does not vary for a given substance so long as the pressure remains the same.*

2. *When cooling the substance solidifies at the same temperature at which it began to melt.*

3. *The temperature at which a given substance melts is always the same for the same pressure.*

The freezing point of water and the melting point of ice are, as we know, 0° C. or 32° F. The following is a table of melting points for a few other substances.

Alcohol	Never Frozen	Lead	326
Mercury	— 38.8° C.	Zinc	425
Sulphuric Acid	— 34.4	Silver (pure)	1000.
Ice	0.	Gold (pure)	1250.
Sulphur	115.	Iron (wrought)	1600.

VAPORIZATION.

If after melting a substance we keep on adding heat until the temperature is very much greater, another change is brought about. The substance from the form of a liquid changes to the form of a gas and during the formation of this gas the temperature remains the same (unless the pressure varies). In other words, sufficient addition of heat overcomes both cohesion and pressure of the atmosphere and the liquid passes into the aeriform condition. This is called vaporization and may be of two kinds, evaporation and ebullition.

Evaporation means the quiet formation of vapor at the free surface of a liquid. The higher the temperature the greater the evaporation; also, the greater the extent of surface the greater the evaporation, and the less the pressure the more rapid the evaporation. A breeze increases evaporation and evaporation is greater in dry air than when it is filled with moisture.

Ebullition or boiling is the rapid formation of vapor in the mass of a liquid. If we place a flask containing water over the flame of a lamp, the air which is contained in the water is driven off in minute bubbles which rise and escape from the surface without noise. As the temperature of the water is increased, the liquid molecules in contact with the bottom of the flask so increase in temperature that the heat overcomes their cohesion, the pressure of the water above and the pressure of the atmosphere. They change into the form of a gas and rise up through the water and escape at the top. At first these bubbles disappear before reaching the surface of the water, because they condense back to water in the cooler layers of water above. This formation and condensation of bubbles of steam produces the sound known as singing

or simmering. When the water becomes sufficiently heated the bubbles increase in number and grow larger as they ascend, break forth at the surface and disappear into the atmosphere. The whole mass of liquid is agitated. There is a certain amount of characteristic noise and the quantity of water diminishes until it is changed completely into steam. In other words the water has been boiled away. Steam is invisible. What is frequently called steam is simply vapor, the minute particles of water arising from condensation of steam, such as the vapor from a kettle. Experiments verify the following law of ebullition.

Every liquid begins to boil at a certain temperature which remains the same until the liquid is entirely vaporized, considering the pressure to remain constant. When cooling the condensation takes place at the same temperature as that at which it changed into the gaseous form.

An increase of pressure raises the boiling point. A decrease of pressure lowers the boiling point. Water, as we know, usually boils at 100° C. or 212° F. under ordinary atmospheric conditions. If we go to the top of a mountain the atmospheric pressure is less because there is not so much air above pressing down on the surface of the liquid, and hence the water boils at a lower temperature. If we close tightly the vessel in which the boiling takes place the pressure increases as the gas is formed, and thus the temperature of boiling is raised.

FIG. 2.

An illustration of this principle is shown by the following experiment. If we heat a vessel of water to boiling, then stop up the opening from which the steam escapes, and remove it from the source of heat, it stops boiling. If now we pour cool water over the top of the vessel, Fig. 2, so that the steam contained in it is condensed back to

water by lowering its temperature, the pressure is decreased by the partial vacuum thus formed, and the water, without any addition of heat, commences to boil again until a sufficient pressure is attained to prevent further ebullition.

We have already noted that substances in the gaseous form occupy far more space than they do in the liquid form. Water in the form of steam occupies nearly 1,700 times as much space as it does in the liquid form, hence a cubic inch of water will occupy when converted into steam nearly a cubic foot of space, the pressure being considered in each case to be one atmosphere. The following is a table of boiling points under a pressure of one atmosphere.

Ammonia	—40°C.	Alcohol	78°C.
Sulphurous anhydride	—8	Water (pure)	100
Ether	35	Mercury	350
Carbon bisulphide	48	Sulphur	447

DISTILLATION.

Distillation is the process of vaporizing a liquid in a heated vessel and then condensing that vapor in a cool vessel. In one

FIG. 3.

vessel the substance changes from a liquid to a gas and in the other vessel, from a gas to a liquid. It is chiefly used for the separation of a liquid from a solid held in solution, or for separating a mixture of two liquids which have different boiling points. The process depends upon the fact that different substances are vaporized at different temperatures.

The apparatus which is called a still has many forms, but consists essentially of two parts, a retort for producing the vaporization and a condenser for condensing the vapor back to the liquid form. This is shown in Fig. 3.

Suppose we wish to separate water from salt which is held in solution. The brine is placed in the retort, Fig. 4, and heated a little above 212° F. At this temperature the water changes to steam, but the salt is not vaporized. The steam passes from the

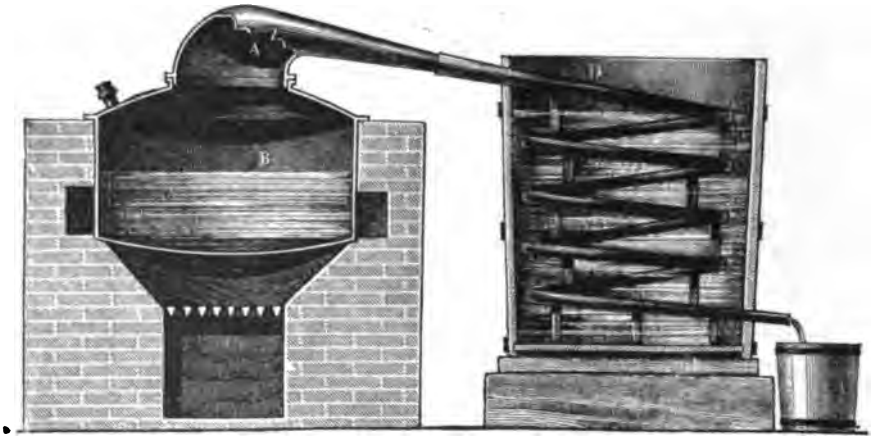


FIG. 4.

retort through a worm where it is condensed rapidly and runs out into a vessel placed to receive it. The salt of course remains in the retort. The water around the worm is kept cool by feeding it with cold water at the bottom and drawing the heated water away at the top.

Another form of apparatus which is generally used in the laboratory is shown in Fig. 5. This consists of a flask for the retort, a tube passing through a water jacket for the worm and another flask, or some other vessel to receive the results of condensation.

Suppose we wished to separate two liquids such as alcohol and water. The retort in which the solution is placed is heated to about 90° C., which is above the boiling point of alcohol but below that of water. The alcohol passes over as vapor while the water

remains behind. In actual practice a certain amount of steam passes over with the alcohol vapor, and the alcohol vapor and steam pass through another vessel surrounded with boiling water, thus

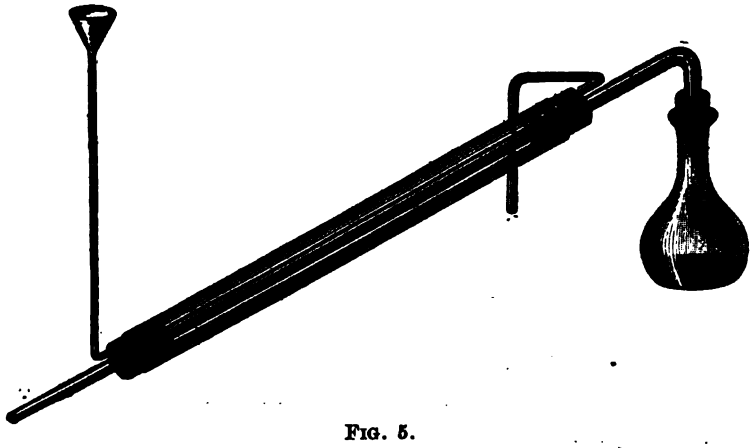


FIG. 5.

condensing out the steam, while the alcohol which does not condense at so high a temperature passes on.

This process is called fractional distillation. In order to obtain alcohol which is very nearly pure, or absolute alcohol as it is called, the distillation is repeated several times.

CONDUCTION.

If we hold one end of a bar of iron in the fire the end which we hold soon becomes hotter than we can bear. This is because it has been heated by conduction, the molecules first heated giving some of their heat to those adjacent and thus passing it on to those beyond, *i. e.*, there is a transfer of motion from molecule to molecule. This process by which heat passes from the hotter to the colder parts of a body is called conduction of heat. It is gradual and does not depend upon the shape of the bar, whether it is crooked or straight.

If we use a rod of glass, the end in the fire would melt but the other end would not become very warm. If we use a wooden stick, the end in the fire would burn and but very little heat would be transferred to the end held. Thus we see that some substances are good conductors of heat, while others are not.

If a bar of iron and one of copper of equal lengths be placed end to end and heated equally at the point of junction, the farther end of the copper bar will become hot before the farther end of the iron bar. The following table gives the relative thermal conductivity of a few metals.

Silver	100	Iron	12
Copper	74	Lead	9
Gold	53	Platinum	8
Brass	24	German silver	6
Tin	15	Bismuth	2

NOTE. The above metals would be arranged in the same order according to the conduction of the electric current, and thus a relation is suggested between heat and electricity.

Liquids and aeriform bodies are poor conductors of heat. Mercury which is a metal is the only fluid which may be readily heated throughout.

CONVECTION.

Water should be heated from below; the heated molecules expand and rise, the cooler ones descend to take their place nearest the source of heat. In this way currents are produced in the water.

Suppose we have a tube whose sides form a rectangle, and a portion of the water in it is colored by some coloring matter. Then if we heat one side of the rectangle the water on that side begins to rise, while the water from the side of the rectangle away from the heat descends and flows across to take the place of the water which has risen. The colored part of the water shows the movement of the current. The same thing takes place in any vessel containing water, the water rising in one part (usually the center) and descending in other parts (usually those parts next to the sides of the vessel). In this way the water becomes heated throughout. This method of diffusing heat by actual motion of heated fluid masses is called convection. It is the result of expansion by heat and the action of the force of gravity.

Convection takes place also in gases; it takes place in the air thereby giving rise to many phenomena, such as wind, rain, etc.

RADIATION.

We have seen that in a heated body the molecules vibrate with an excess of rapidity. The motion of these vibrating molecules is communicated to the ether, and transmitted by it with great velocity in the form of waves. If near a fire we feel the heat from it and heat thus propagated by the ether, instead of by material means, is called radiant heat, while the process is called radiation. In this way the sun transmits heat to the earth.

That radiant heat traverses a vacuum may be shown experimentally. Radiant heat travels in straight lines through a uniform medium and the radiation is equal in all directions. Thermometers placed at equal distances from a source of heat show the same temperature. The amount of radiation depends upon the temperature of the source and is inversely proportional to the square of the distance. Hence a differential thermometer at a distance of 2 feet would only show $\frac{1}{4}$ the radiation that it would at 1 foot.

If heat rays strike a surface they may be transmitted, absorbed or reflected. If they are transmitted they may be refracted, or bent, as light is by means of a lens, Fig. 6. Rock salt crystals

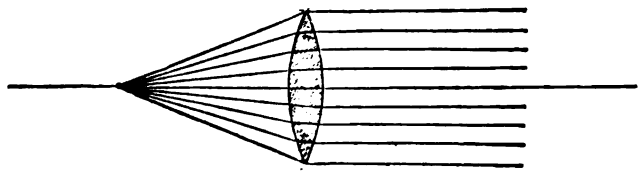


FIG. 6.

transmit nearly all the rays, absorb very little and reflect very little. Polished silver reflects nearly all, absorbs a little and transmits none. Charcoal absorbs nearly all, and hence increases in temperature quite rapidly when under the influence of heat rays.

Bodies that transmit radiant heat freely are called diathermanous, and those that do not are called athermanous. These terms are to heat what transparent and opaque are to light. Heat may also be reflected, the angle of incidence equaling the angle of reflection, or in other words, if a perpendicular be drawn to the surface at the point where the heat ray strikes, this heat ray will

make the same angle with the perpendicular before reflection that it will after reflection. If parallel rays, such as those of the sun, strike upon a spherical or parabolic mirror they are reflected to a focus. Suppose that the mirror at the right, Fig. 7, has a source of heat at its focus. Then it will be found that the ball at the focus of the left hand mirror will be heated by reflected rays.

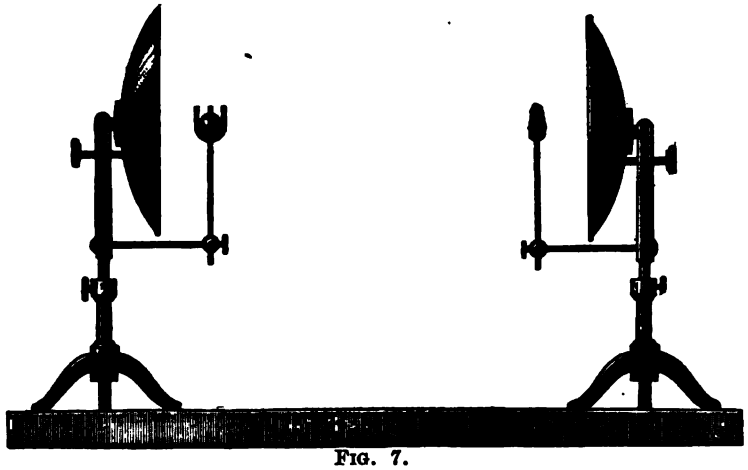


FIG. 7.

Suppose a body receives 10 units of heat and is at a distance of 6 feet from the source. How may we find the amount of heat received at a distance of 12 feet? Since the heat radiated is inversely proportional to the square of the distance, the amount of heat received at a distance of 12 feet must be $\frac{1}{4}$ that received at a distance of 6 feet. Expressed as an equation we have,—

$$10 : x = \frac{1}{6^2} : \frac{1}{12^2}$$

or $$x = 2\tfrac{1}{2}$$

Hence the heat received at a distance of 12 feet is $2\frac{1}{2}$ units.

EXAMPLES FOR PRACTICE.

1. If a body receives 8 heat units at a distance of 6 yards from the source, how many heat units will it receive at a distance of 24 yards? Ans. $\frac{1}{2}$ heat unit.

2. A body shows a temperature of 80° C. when at a distance of 2 feet from a source of heat; what will be its temperature dependent upon radiant heat received at a distance of 4 feet from the source? Ans. 20° C.

LATENT HEAT.

The units used in measuring heat are called thermal units. The one most used in this country is called the British thermal unit. It is the amount of heat necessary to raise 1 pound of water at maximum density (39.2° F.), one degree F. in temperature.

In the French system we have the gram-calorie which is the amount of heat necessary to raise 1 gram of water from 4° to 5° C. and the kilogram-calorie, which is 1000 times as great, is the amount of heat necessary to raise a kilogram of water from 4° to 5° C. It does not make any practical difference what unit is used; it is a matter of convenience, but the unit must not be changed in the course of any calculation.

Suppose, now, that we take a block of ice at — 10° C. (14° F.) and heat it. A thermometer placed in it rises to 0° C. The ice then begins to melt, but the thermometer does not indicate any further rise in temperature. It remains at 0° C. until the ice is melted and then begins to rise again. If we continue to apply heat, the water rises in temperature until it reaches 100° C. (212° F.) when the water begins to change into steam and the thermometer shows no further rise. By this we see that we have applied a very considerable amount of heat without producing any rise in temperature. This heat which does not produce any rise in temperature is called latent (the word meaning hidden) heat.

Latent heat may be defined as the heat that must be added to a body in a given state to change it into another state without altering its temperature. The latent heat which is required to melt ice, or any other solid, is called the latent heat of fusion. In the case of ice we may represent the action of latent heat by the following equation.

Water at 0° C. = ice at 0° C. + latent heat of water.

If we dissolve a solid, *i. e.*, put it into solution, there is a diminution of temperature and this diminution depends upon the fact that in the process of solution as well as in fusion, heat is rendered latent. This latent heat of solution is the basis of the action of freezing mixtures. Thus when ice is melted by salt the water formed dissolves the salt itself and this double liquefaction, or the melting of the ice and the dissolving of the salt, abstracts a

certain amount of heat from whatever may be in its proximity. In the ice-cream freezer the heat is taken from the cream, thus lowering its temperature and causing it to freeze. A freezing mixture of 1 part salt and 2 parts snow by weight gives a temperature of about — 18° C. and furnishes the zero adopted by Fahrenheit.

Solidification takes place when a liquid changes to the solid condition. During solidification we have an increase in heat. Sensible heat which disappeared as latent heat during liquefaction is no longer employed in maintaining a liquid state, and is therefore re-converted and is immediately employed in increasing the molecular vibrations; in other words the molecular potential energy is transformed into molecular kinetic energy. The melting point and the solidifying point of the solid substance are identical, but it is possible to keep substances in a liquid condition at somewhat lower temperatures. If water is perfectly quiet it sometimes cools several degrees below the melting point without freezing, but the slightest agitation, such as tapping on the vessel in which it is contained, causes solidification to take place immediately. When this takes place the temperature rises instantly to the freezing point. Most substances shrink when they solidify but a few, among which are ice and cast iron, expand upon solidification. Antimony and bismuth expand, and for this reason antimony is used as one of the constituents of type metal. Gold coins have to be stamped, and clear cut castings cannot be obtained from lead because these metals contract when they solidify. Water expands considerably when changing to ice. This is apparent from the fact that vessels filled with water will be broken if exposed to a low temperature.

When a liquid is vaporized there is, as we know, a disappearance of a large quantity of heat, and frequently a diminution in temperature. There is also a change of sensible into latent heat; of kinetic into potential energy. We may represent a simple case by the following equation.

Steam at 100° C. = water at 100° C. + latent heat of steam.

Let us now see how great an amount of heat is latent in the cases of fusion and vaporization of water. If we mix a pound of water at 0° C. with a pound of water at 80° C. we have as a result 2 pounds of water at 40° C., but if we mix a pound of ice at 0° C.

with a pound of water at 80° C. we have 2 pounds of water at 0° C. Hence, the heat which before was used to raise the temperature of the water has been used in melting the ice. The amount of heat required to melt a quantity of ice without changing its temperature is 80 times as great as the heat required to warm the same quantity of water 1° C.

In the case of vaporization of water, the amount of heat necessary to evaporate a weight unit of water is the amount sufficient to raise the temperature of 537 weight units 1° C. In other words, the amount of heat necessary to evaporate a certain quantity of water without changing its temperature is 537 times as great as the heat required to raise the same quantity of water 1° C.

If instead of the above method we use the English system, then the amount of heat necessary to convert a pound of water at 212° into steam at 212° is 966 British thermal units, or sufficient heat to raise 966 pounds of water 1° F. (the water being at maximum density and the pressure one atmosphere). The total heat of evaporation for a given temperature is the amount of heat which is added to change water at freezing, 32° F., into steam at the given temperature. It is the sum of the sensible and latent heats at that temperature.

Let us see what the total heat of evaporation is for 1 pound of water at freezing, to steam at boiling temperature and atmospheric pressure. The sensible heat is $212 - 32 = 180$ British thermal units, and the amount of latent heat to change the water at 212° into steam at atmospheric pressure is 966 British thermal units. Hence, the total heat applied is $180 + 966 = 1146$ British thermal units, which is the total heat of evaporation in the above case. The following table gives the sensible, latent and total heats of evaporation for 1 pound of water at various pressures.

Temperature of Boiling Water	Pressure in lbs. per Square Inch	Sensible Heat in Thermal Units	Latent Heat in Thermal Units	Total Heat in Thermal Units
212°	15	180	966	1146
306	75	274	900	1174
339	120	307	876	1183
357	150	325	862	1187

We see that in the last column the figures do not vary to any

great extent. In the second row where the absolute pressure is 75 pounds, *i. e.*, the atmospheric pressure + 60 pounds, the total heat of evaporation is 1174 B. T. U. In the 4th row the pressure is at 150 pounds. The number of B. T. U. is 1187. From this we see that by the addition of only 13 B. T. U. to each pound of boiling water at 75 pounds, the steam pressure is doubled. This fact is taken advantage of in modern marine engines where a very high pressure is used and the steam is allowed to expand.

EXAMPLES FOR PRACTICE.

1. One kilogram of water at 40° C., 2 kilograms at 30° C., 3 kilograms at 20° C., and 4 kilograms at 10° C. are mixed. Find the temperature of the mixture. Ans. 20° C.

2. What weight of water at 85° C. will just melt 15 pounds of ice at 0° C.? Ans. 14.117 + pounds.

3. One pound of ice at 0° C. is placed in 5 pounds of water at 12° C. What will be the result?

Ans. $5\frac{3}{4}$ pounds of water at 0° C. and $\frac{1}{4}$ pound of ice at 0° C.

SPECIFIC HEAT.

If the temperature of a body changes from 30° to 20° the body loses the same amount of heat that it gained in changing from 20° to 30°. We may measure the heat lost by a cooling body just as we may measure any other form of energy, by the work it can perform. Equal weights of different bodies at the same temperature will melt amounts of ice proportional to the number of thermal units contained in each.

A pound of sulphur at 212° F. melts $\frac{1}{5}$ as much ice as a pound of boiling water. Hence, we see that it requires $\frac{1}{5}$ as much heat to raise sulphur from the freezing point to 212° F. as it does to heat the water to that same temperature. In other words, the specific heat of sulphur is $\frac{1}{5}$. The specific heat of a body is the ratio between the quantity of heat required to warm that body 1° and the quantity of heat required to warm an equal weight of water 1°. Specific heat like specific gravity is a ratio. Water in its liquid form has a higher specific heat than any other substance except hydrogen. Thus the specific heat of water is taken as the standard or unit.

One of the simplest methods of determining specific heat is by mixing the substance with water. Suppose that 6 pounds of mercury at 100° C. are mixed with 2 pounds of water at 0° C., the temperature of the resultant mixture registers 9° C., and we wish to find the specific heat of mercury. The number of heat units lost by the mercury in passing from 100° to 9° C. is equal to the weight taken, times the number of degrees of change, times its specific heat. Let x represent the specific heat. Then the number of heat units lost by the mercury is $6 \times 91 \times x$ or $546\ x$. Now the specific heat of water is taken as 1, because it is our standard or unity. The number of heat units gained by the water will be the number of degrees change, times the weight taken, times its specific heat, or $2 \times 9 \times 1$ or 18 heat units This number of heat units which is gained by the water is equal to the number of heat units lost by the mercury or;

$546\ x = 18$ and

x the specific heat of mercury $= \frac{9}{273} = .033$

The following is a table of specific heats of various substances, (gases and vapors under constant pressure):

Hydrogen	3.4090	Iron (wrought)	.1138
Water	1.0000	Copper	.0952
Ammonia (gas)	.5084	Silver	.0570
Air	.2375	Tin	.0562
Oxygen	.2175	Mercury	.0333
Sulphur	.2026	Lead	.0314
Diamond	.1469	Bismuth	.0308
Cast Iron	.1298	Glass	.1937
Steel (hard)	.1175	Zinc	.0956
Steel (soft)	.1165	Gold	.0324
Nitrogen	.2438	Ether (gas)	.4797
Carbon monoxide	.2479	Chloroform	.1567
Carbon dioxide	.2170	Ice	.5050

Let us find the weight of water at 95° C. that will just melt 10 pounds of ice at — 10° C. The latent heat consumed in melting 10 pounds of ice at 0° C. is 80×10, or 800 heat units, but the ice must first be raised from — 10° C. to 0° C. Now the heat necessary to raise the ice to 0° C. is $10 \times 10 \times$ the specific heat of ice, or $10 \times 10 \times .505 = 50.5$. Therefore $800 + 50.5 =$

850.5 heat units are necessary to melt the ice. We know that one pound of water at 95° C. will give up 95 heat units if lowered to 0° C. Hence the weight necessary to furnish 850.5 heat units must be 850.5 ÷ 95 or 8.95 + pounds.

EXAMPLE FOR PRACTICE.

1. What weight of water will melt 4 pounds of ice at—12° C. if the water used is at 60° C.? Ans. 5.74 pounds (nearly).

THERMODYNAMICS.

Thermodynamics is the branch of science which considers the relation between heat and mechanical work. It deals with the numerical relation between the quantity of heat used and the amount of work done.

We have seen that heat is a form of energy. If we hammer a piece of metal we find that it becomes hot. This is because the mechanical energy of hammering is converted into heat. Matches are ignited by the heat generated by friction; an instance of the conversion of mechanical energy into heat. In the case of journals and axles we try to prevent this conversion by the use of lubricants which decrease the coefficient of friction. The friction which appears as heat forms a part of the resistance to the pull of the locomotive. A railway train is really stopped by the conversion of its energy of motion into the form of heat, the action of the brakes being to increase the friction thus causing more heat to be developed.

The first law of thermodynamics may now be stated. It is as follows:

When heat is transformed into mechanical energy, or mechanical energy into heat, the quantity of heat is equivalent to the quantity of mechanical energy.

Dr. Joule of Manchester, England, was the first to ascertain the relation numerically. After various experiments he obtained the following results:

The mechanical value of the heat required to warm a given weight of water, 1° C., would lift the water 424 meters or 1390 feet

against gravity. The heat required to warm it 1° F., would lift the water 772 feet against gravity. In other words the amount of heat necessary to warm a kilogram of water 1° C., would, if expended in mechanical effort, lift that weight of water 424 meters, or 1,390 feet, while acting against the force of gravity, *i. e.*, that amount of heat is equivalent to 424 kilogrammeters. If the temperature is expressed in degrees F. the heat necessary would raise the weight of water 772 feet against gravity, *i. e.*, the heat necessary to raise a pound of water 1° F. is equivalent to 772 foot-pounds. We might have said the heat necessary to raise a pound of water 1° C. expressed in mechanical effort would raise 424 pounds 1 meter or 1390 pounds 1 foot against gravity. Also if the temperature were taken in F. degrees the heat would raise 772 pounds 1 foot against gravity.

These values of the mechanical equivalent of heat are based upon the results of Joule's experiments in 1843–50. Later experiments have shown that these values are too small. Prof. Rowland gives higher figures and the most probable average is now considered to be 778. Hence the British thermal unit is equivalent to 778 foot-pounds of energy. To express this relation in French units we say that the mechanical equivalent of heat would raise 427 kilograms 1 meter against gravity. This equivalent is called the calorie equivalent and is equal to 427 kilogrammeters.

Let us repeat these values that they may be firmly fixed in the mind. The British thermal unit is the amount of heat required to raise the temperature of 1 pound of water 1 degree Fahrenheit, and is equivalent to 778 foot-pounds of mechanical energy. The French calorie is the amount of heat required to raise the temperature of 1 kilogram of water 1 degree Centigrade, and is equivalent to 427 kilogrammeters of mechanical energy.

The previous values it must be noticed are average. The equivalent will vary considerably, dependent upon the temperature at which it is found, as well as upon the value of *g*, *i. e.*, the locality. The following table gives the number of kilogrammeters required to raise one kilogram of pure water one degree from the specified temperature, as found by Prof. Rowland, for the latitude of Baltimore, and at sea level.

Temperature Centigrade.	Equivalent in Kilogrammeters.	Temperature Centigrade.	Equivalent in Kilogrammeters.
0°	431.1	23°	426.0
5	429.8	25	425.8
10	428.5	27	425.6
15	427.4	29	425.5
17	427.0	31	425.6
19	426.6	33	425.7
21	426.2	35	425.8

The values given in the table may be reduced for other latitudes than Baltimore by adding for; lat. 30° + 0.34 kgm.; lat. 40° + 0.08 kgm.; lat. 50° — 0.41 kgm.; lat. 60° — .77 kgm. To reduce to foot-pounds and degrees Fahrenheit, multiply by 1.8227.

Let us find the temperature to which a cannon ball weighing 192.96 pounds and moving with a velocity of 2000 feet per second would heat 100 pounds of ice-cold water. Suppose the cannon ball to be suddenly stopped and all its kinetic energy converted into heat. Its kinetic energy is:

$$\frac{w\,v^2}{2\,g} = \frac{192.96 \times 4000000}{64.32} = 12{,}000{,}000 \text{ foot-pounds.}$$

12,000,000 ÷ 778 = 15,424 + heat units. Since 15,424 heat units are expended in raising 100 pounds of water, the amount of heat available per pound is 15,424 ÷ 100 = 154.24. Hence the water would be raised to 154.24 + 32 = 186.24° F.

EXAMPLES FOR PRACTICE.

1. A certain source of heat raises the temperature of 10 pounds of water from 36° F. to 90° F. Find the necessary mechanical equivalent in foot-pounds. Ans. 420120 foot-pounds.

NOTE. Use 778 as the mechanical equivalent of heat.

2. Find the value of the equivalent in foot-pounds, Fahrenheit degrees, latitude 60°, if the water used is at a temperature of 31° C. Ans. 774.3 + foot-pounds.

APPENDIX

QUESTIONS IN REVIEW

The following lists of questions afford the reader a means of testing his knowledge of the subjects treated in the present volume. Any one who has carefully read the foregoing subjects should be able to answer every question accurately. The reader is advised to go through the questions thoroughly, working out his answers and checking his results with the answers given. This will afford a valuable means of fixing the matter in mind. Students who are preparing for examinations for college, civil service or engineer's license will find the questions of the greatest value.

REVIEW QUESTIONS

ON THE SUBJECT OF

GEOMETRY.

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1. What do you mean by *quantity?*
2. What dimensions has a solid? A line? A point?
3. What is an angle?
4. What is the sum of the angles about a point? On one side of a straight line?
5. What are parallel lines?
6. What is the sum of all three angles of any triangle? Prove it.
7. Under what conditions are triangles equal?
8. What is an isosceles triangle?
9. What is a parallelogram?
10. How is the area of a rectangle found? That of a parallelogram?
11. How do you find the area of a triangle? The base of a triangle is 5 inches, the altitude 7 inches. What is its area?
12. What does the square of the hypothenuse of a right angle equal?
13. The angle at the center of a circle varies as what?
14. By what is an inscribed angle measured?
15. How do you find the area of a circle? The radius of a circle is 3 inches. What is its area?
16. How would you bisect a straight line?
17. How would you draw a perpendicular to a straight line?
18. How would you bisect an arc?
19. How would you construct a triangle, the three sides being given?
20. How would you find the center of a circle?
21. How would you inscribe a square in a circle? A hexagon?
22. What is meant by a plane?

REVIEW QUESTIONS

ON THE SUBJECT OF

TRIGONOMETRY AND LOGARITHMS.

EXERCISE I.

1. Find the values of the remaining five functions when $\sin A = \frac{1}{3}$.

2. Find the values of the remaining five functions when $\sec A = 5$.

3. Add $117° 23' 26''$ and $42° 46' 35''$.
   From $59° 12' 40''$ take $72° 28' 32''$.

4. Show that if A and B are the acute angles of a right triangle, $\sin A = \cos B$ and $\cot A = \tan B$.

EXERCISE II.

1. By using formulas [3] to [8] show that $\tan A + \cot A = \sec A \csc A$.

2. Find $\sin 225°$, $\tan 150°$.

3. By method of Art. 56 find $\log_5 125$.

4. Find $\log \cos 118°$, $\cot 167°$.

EXERCISE III.

Find by the use of logarithms the value of

1. 3.1425×603.93.

2. $\frac{3^{\frac{1}{2}}}{8^{\frac{1}{3}}}$.

3. $(.0136)^2 \div \sqrt[5]{.00576}$. Ans. .00051879.

4. Log $\tan 67° 13' 27''$.

5. Log $\cot x = 0.282956$, find x.

EXERCISE IV.

Solve the right triangles:

1. Given B = 75° 12′, c = 14.67.
2. Given a = 273, b = 418.
3. Given A = 33° 47′ 16″, a = .489.

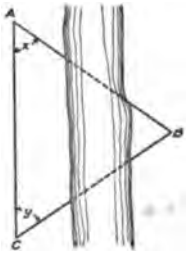
Ans. $\begin{cases} b = .73080 \\ c = .87931 \\ \text{B} = 56° 12′ 44″. \end{cases}$

Solve the oblique triangles:

4. Given b = 6.074, A = 70° 39′, B = 56° 23′.
5. Given a = 63.49, c = 49.13, C = 37° 14′ 12″.

EXERCISE V.

1. To find the horizontal distance from a point A to an inaccessible point B, the horizontal distance A C and angles x and y were measured and found to be 1,042.3 feet, 72° 9′ and 14° 13.7′ respectively. Find A B and C B.



Ans. $\begin{cases} \text{A B} = 256.7 \text{ ft.} \\ \text{C B} = 994.11 \text{ ft.} \end{cases}$

2. The angle of elevation of the top of a tower observed from a point at a horizontal distance of 897.3 ft. from the base is 10° 27′ 42″. Find the height of the tower.

Ans. 165.68+ ft.

NOTE.—The angle of elevation or depression of an object is the angle a line drawn from the point of observation to the object makes with the horizontal.

3. Find the area of a triangular field, the sides of which are 976.5 ft., 837.2 ft. and 1,342 ft. respectively.

Ans. 407,119 sq. ft.

NOTE.—Use Formula [37].

REVIEW QUESTIONS

ON THE SUBJECT OF

MECHANICS.

EXERCISE I.

1. Define (*a*) molecule, (*b*) atom.

2. What are the three forms in which matter may exist?

3. Distinguish between a gas and a liquid.

4. (*a*) Name the universal properties of matter, and (*b*) what is a characteristic property of matter?

5. What is meant by tenacity?

6. What is meant by inertia?

7. What is porosity?

EXERCISE II.

1. Define (*a*) motion, (*b*) velocity.

2. Distinguish between uniform and variable velocity.

3. A rolling ball has a uniform velocity. It travels 2,400 feet in 8 minutes. What is its velocity in feet per second?

Ans. 5 feet per second.

4. Define variable velocity, and give an example of the same.

EXERCISE III.

1. Give a definition of force. In treating of forces what three things must be considered?

2. What does the momentum of a body measure or signify?

REVIEW QUESTIONS

ON THE SUBJECT OF

MECHANICS.

EXERCISE I.

1. Why does water cling to the hand when it is dipped?

2. What is adhesion?

3. Is capillary attraction a result of cohesion or of adhesion?

4. What substance with which you are familiar exerts strong capillarity?

5. What is cohesion?

6. How may we entirely destroy the cohesion of a liquid?

EXERCISE II.

1. (a) When is a body stressed? (b) What is the difference between *tensile* and *compressive* stress?

2. (a) Define unit stress. (b) Unit deformation.

3. What is the unit elongation per foot of a block six feet long if the total elongation is .012 inch? Ans. .002 inch.

4. State as fully as you can the action of a piece of iron that is loaded beyond its elastic limit.

5. (a) Define ultimate breaking strength. (b) Permanent set. (c) What can you say of the elasticity of a body which is subjected to a small stress?

6. What is the relation of the deformation of a body under small stress to its length?

7. What is the ultimate tensile strength of steel?

8. Find the coefficient of elasticity of a substance, if the unit stress is 48,000 pounds and the unit deformation is .0016.

Ans. 30,000,000.

9. Find the area of a square wrought-iron bar that will just stand a tensile stress of fifty-five tons. The breaking stress is 50,000 pounds per square inch. Ans. $2\frac{1}{8}$ square inches.

10. What is the elongation of a bar 18 inches square and 24 feet long, under a tensile stress of 9,000 pounds, if the unit elongation is .007 inch? Ans. .168 inch.

11. A bar of steel is $\frac{7}{8}$ inch in diameter and 6 inches long. If it breaks under a load of 45,000 pounds, what is the unit breaking strength? Ans. 75,000 pounds (about).

REVIEW QUESTIONS

ON THE SUBJECT OF

HEAT.

EXERCISE I.

1. A body has a temperature of 512° F.; what do you know with regard to the amount of heat the body possesses?

2. Give as good a definition as you can in your own words of the term "cold."

3. How is the zero point of the Fahrenheit thermometer determined?

4. Express a temperature of —64° F. in Centigrade degrees.

Ans. $-53\frac{1}{9}$° C.

5. How many degrees F. would a temperature of —45° C. indicate?

6. Give an example of linear expansion from your own observation.

7. At what temperature in degrees Centigrade and Fahrenheit is water at its maximum density?

8. Find the length of a wrought-iron piston-rod when it has been heated to 140° F. Its length at 40° F. is 6 feet.

Ans. 6.003888 feet.

EXERCISE II.

1. Find the volume of 720 liters of air at 37° C., when heated to 90° C. Ans. 843 liters (approx.).

2. A cylinder holds 12 cubic feet of CO_2 gas at 100° C., and for every cubic foot of expansion the pressure is increased 4 pounds. What must be the increase of pressure if the gas is heated to 110° C.? Ans. 1.2 + pounds.

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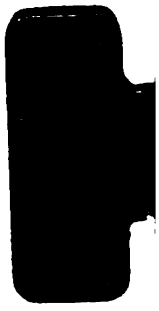

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